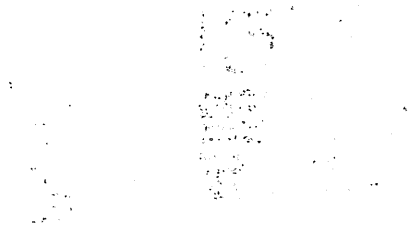
INTRODUCTION TO QUANTUM FIELD THEORY



QC
74.45
C445
990

Lecture Notes in Physics – Vol. 29

INTRODUCTION TO QUANTUM FIELD THEORY

S. J. Chang
University of Illinois at Urbana-Champaign
Illinois, USA

World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 9128

USA office: 687 Hartwell Street, Teaneck, NJ 07666

UK office: 73 Lynton Mead, Totteridge, London N20 8DH

Library of Congress Cataloging-in-Publication Data

Chang, S.-J.

Introduction to quantum field theory/by S. J. Chang.

(World scientific lecture notes in physics, 0218026X; vol. 29)

ISBN 9971506270

9971506815 (pbk)

ISSN 0218-026X

1. Quantum field theory. I. Title. II. Series.

QC174.45.C445 1990

530.1'43--dc20 89-77616

CIP

Copyright © 1990 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the Publisher.

Printed in Singapore by JBW Printers & Binders Pte. Ltd.

PREFACE

This book originates from a set of lecture notes for a two-semester graduate course in quantum field theory which the author taught at the University of Illinois. The purpose of the first part of the book is to teach a graduate student the basic skill of perturbation theory in both particle physics and in many-body physics. We expect that the student has a general background in nonrelativistic quantum mechanics. We begin with a quantum system with a finite number of particles and then generalize it to the quantum field. We follow Dyson-Wick's approach, and introduce Feynman diagrams. We apply the diagrammatic technique to quantum electrodynamics and to zero-temperature many-body system.

In the second part of the book, we begin with Schwinger's action principle. Schwinger's action principle is probably the most elegant method of introducing field quantization. The action principle is applicable with ease to both boson fields and fermion fields. We illustrate how to construct stress tensors and how to verify Lorentz invariance in the framework of quantum field theory. Next we develop Feynman's path integral method which is complementary to Schwinger's differential method. Path integral formulation provides the most efficient way of deriving Feynman rules for non-abelian gauge theories. A serious theory student should master both methods. The remaining part of the book is devoted to renormalization theory including the renormalization group method.

The following excellent books provide additional reading and references:

1. J. Schwinger, ed., *Quantum Electrodynamics*, Dover , New York, 1958.
2. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, New York, 1964, and *Relativistic Quantum Fields*, McGraw-Hill, New York, 1965.
3. R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.
4. A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle*

Systems, McGraw-Hill, New York, 1971.
5. C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill, New York, 1971.
6. D. J. Amit, *Field Theory, the Renormalization Group, and Critical Phenomena*, World Scientific, Singapore, 1984.

I wish to thank Professor K. K. Phua for suggesting to turn my lecture notes into a book, and for his patience and encouragement. World Scientific has done a beautiful job in transforming the typed lecture notes into its present form. I wish to thank Mary Ostendorf and Sue White for their help in preparing the draft version of the lecture notes.

Shau-Jin Chang
Illinois
February, 1990

NOTATIONS

In this book, we use the metric tensor

$$g_{\mu\nu} = (1, -1, -1, -1) \ .$$

As space-time indices, all greek letters take values from 0 to 3, and all latin letters take values from 1 to 3. Repeated indices are to be summed over,

$$A_\mu B_\mu \equiv A_\mu B^\mu = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3 \ ,$$

$$A_i B_i \equiv A_1 B_1 + A_2 B_2 + A_3 B_3 \ .$$

When we encounter both the group and the space-time indices, we keep the group indices as superscripts and the space-time indices as subscripts. We denote the group indices by letters a, b, c, etc.

We choose our units such that $c = \hbar = 1$.

CONTENTS

Preface v

Notations vi

Part I GENERAL FIELD THEORY

Chapter 1 MANY PARTICLE SYSTEMS AND FIELD THEORIES 3
1. An NR Quantum System with n Degrees of Freedom 3
2. Continuum Limit 4
3. A Free Field 6
4. Nonrelativistic Many-particle Systems 13

Chapter 2 RELATIVISTIC FREE FIELD THEORIES 16
1. Free Scalar Fields 16
2. Free Dirac Equations 20
3. Plane Wave Solutions to Dirac Equations 26
4. Free Dirac Fields 29
5. Negative Energy Sea 32

Chapter 3 INTERACTIONS AMONG FIELDS 36
1. An Example of Field Interactions 36
2. Interaction Picture and Dyson's Expansion 38
3. S-Matrix, In-states, and Out-states 42
4. Wick's Theorem 43
5. Second Order Processes 48
6. Explicit Amplitudes 53

Chapter 4 FEYNMAN DIAGRAMS AND CROSS SECTIONS 58
1. Position-space Feynman Rules 58
2. Momentum-space Feynman Rules 59
3. Cross Sections 64
4. Phase-space Factors 67
5. An Example 68

Chapter 5 QUANTUM ELECTRODYNAMICS 74
1. The Classical Electromagnetic Field 74
2. Quantization of the Electromagnetic Field 78
3. The Photon Propagators 80
4. The Minimal Coupling and the Feynman Rules 82
5. Electron-electron Scattering 85
6. Interactions with the Classical Sources 89
7. Compton Scattering and the Low Energy Theorem 95

Chapter 6 HIGHER ORDER PROCESSES 98
1. Degrees of Divergence 98
2. The Vacuum Polarization Diagrams 99
3. Electron Self-energy Diagrams 109
4. Vertex Corrections 112

Chapter 7 ZERO TEMPERATURE FERMI SYSTEM 121
1. Introduction 121
2. The Feynman Rules 122
3. Hartree-Fock Approximation 126

Chapter 8 THE ELECTRON GAS MODEL 130
1. The Model 130
2. Hartree-Fock Approximation 131
3. Polarization and Screening 134
4. Decay of Electrons and Holes 139

Chapter 9 BRUEKNER THEORY OF NUCLEAR MATTER 145
1. Nuclear Matter 145
2. Bruekner Theory 146

Chapter 10 ZERO TEMPERATURE BOSE SYSTEM 153
1. Noninteracting Bose System 153
2. Zero Temperature Interacting Bose System 156
3. Self-energy Diagrams 160

Chapter 11 LIQUID HELIUM-4 165
1. Review of Facts 165
2. Superfluidity 168

Chapter 12 CLASSICAL SYSTEMS WITH NOISES 172
1. Equivalent Syetems 172
2. Feynman Rules 177
3. Fokker-Planck Equations and Boltzmann Distribution 180

Part II ADVANCED FIELD THEORY

Chapter 1 FIELD QUANTIZATIONS 187
1. Action Principle 187
2. Quantization of a Scalar Field 191
3. Quantization of a Dirac Field 193
4. Quantization of a Vector Field 196

Chapter 2 LORENTZ TRANSFORMATION PROPERTIES 202
1. Lorentz Invariance 202
2. Free Scalar and Dirac Fields 205
3. Stress Tensor 211
4. Schwinger Relation 215

Chapter 3 DISCRETE TRANSFORMATIONS 218
1. Parity 218
2. Charge Conjugation 220
3. Time Reversal 223
4. TCP Theorem 226
5. Spin and Statistics 227

Chapter 4 PATH INTEGRAL FORMULATION 230
1. Nonrelativistic Quantum Mechanics 230
2. Classical Limit 232
3. Hamiltonian Path Integrals 238
4. Scalar Field Theories and Feynman Rules 240
5. Dyson-Schwinger Equation 247
6. Grassmann Algebra 250
7. Fermion System 253

Chapter 5 GAUGE THEORIES 258
1. Path Integral Formulation of Maxwell Fields 258
2. Quantum Electrodynamics 265
3. Yang-Mills Fields 268
4. Path Integrals and Feynman Rules 272
5. Examples 278

Chapter 6 RENORMALIZATION THEORY 281
1. Renormalization of QED — One Loop Diagrams 289
2. Dyson's Prescriptions 286
3. Overlapping Divergences 290
4. BPHZ Renormalization 292
5. Classification of Renormalizable Theories 297

Chapter 7 RENORMALIZATION OF NON-ABELIAN GAUGE THEORIES 300
1. Non-abelian Gauge Theories 300
2. Gauge Field Self-energy Diagrams 303
3. Other Renormalization Parts 310
4. Slavnov-Taylor Identities 313
5. Higgs Mechanism 318

Chapter 8 RENORMALIZATION GROUP 322
1. Intermediate Renormalization and its Group Property 322
2. Callan-Symanzik Equations 326
3. Solutions to Callan-Symanzik Equations 330
4. An Example 333
5. Asymptotically Free Theories 338

Chapter 9 INSTANTON 343
1. Solutions to a Classical Yang-Mills Theory 343
2. Instanton as a Tunneling Solution 347
3. Tunneling Potential and Vacuum 352
4. Zero Modes 357
5. Axial Anomaly 364
6. Decay of a False Vacuum 368

Subject Index 373

Part I

GENERAL FIELD THEORY

Chapter 1

MANY PARTICLE SYSTEMS AND FIELD THEORIES

In this chapter, we begin with a nonrelativistic (NR) quantum mechanical system with n degrees of freedom. We then generalize it to the continuum limit. This will lead naturally to the field quantization.

1. An NR Quantum System with n Degrees of Freedom

Consider an n degrees of freedom quantum system

$$H(q, p) = \Sigma\, p_i^2/2m + V(q). \tag{1.1}$$

The coordinates q_i and the momenta p_i obey the equal-time quantization relations

$$[q_i, q_j] = [p_i, p_j] = 0 \tag{1.2a}$$

$$[q_i, p_j] = i\delta_{ij}\hbar. \tag{1.2b}$$

In the following, we shall choose the units such that $\hbar = 1$. Equations (1.1) and (1.2) determine the system completely. We can describe the system either in the Schrödinger picture, or in the Heisenberg picture. In the Schrödinger picture, the time evolution of the physical system is in the wave function $\psi(q, t)$ which satisfies the Schrödinger equation,

$$i\frac{\partial\psi(q, t)}{\partial t} = \left[-\frac{1}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right]\psi(q, t). \tag{1.3}$$

In the Heisenberg picture, the time evolution is associated with the operators $p_i(t)$ and $q_i(t)$ which obey the Heisenberg equations of motion,

$$\dot{q}_i = \frac{1}{i}[q_i, H] = p_i/m \tag{1.4a}$$

$$\dot{p}_i = \frac{1}{i}[p_i, H] = -\frac{\partial V}{\partial q_i}. \tag{1.4b}$$

We can establish easily that these two representations are equivalent.

2. Continuum Limit

Next we consider a string of density ρ, under tension T, and lying near the bottom of a cylinder, as shown in Fig. 1.1a.

Classically, we may describe the dynamics of the string by its transverse displacement $q(x, t)$ from its equilibrium position at (x, t) (See Fig. 1.1b). Note that the continuous variable x is the analog of the discrete index i discussed earlier. Thus, x is an index parameter, and not a dynamical variable. The dynamical variables are $q(x, t)$ and $\dot{q}(x, t)$ which are the generalization of q and p/m. Quantities $q(x, t)$ and $\dot{q}(x, t)$ are simple examples of field variables.

Classically, the generalization of a discrete system to a continuum system is straightforward. The Lagrangian and Hamiltonian of this string system are

$$L = \int_0^L dx\, \mathscr{L}(q, \dot{q}) \tag{1.5}$$

$$H = \int_0^L dx\, \mathscr{H}(q, \pi) \tag{1.6}$$

where

$$\mathscr{L} = \frac{1}{2}\left[\rho\dot{q}^2 - T\left(\frac{\partial q}{\partial x}\right)^2\right] - V(q) \tag{1.7}$$

$$\mathscr{H} = \pi^2/2\rho + \frac{T}{2}\left(\frac{\partial q}{\partial x}\right)^2 + V(q) \tag{1.8}$$

$$\pi \equiv \frac{\partial \mathscr{L}}{\partial \dot{q}} = \rho\dot{q}\,. \tag{1.9}$$

In Eqs. (1.7) and (1.8), $V(q)$ is the potential energy density of the string as it moves along the cylinder. (See Fig. 1.1c.)

When we treat the string as a quantum system, we need to impose the equal-time quantization rules

$$[q(x, t), q(y, t)] = [\pi(x, t), \pi(y, t)] = 0 \tag{1.10}$$

$$[q(x, t), \pi(y, t)] = i\delta(x - y). \tag{1.11}$$

In short, to change from a discrete system to a continuum system, we need to replace discrete summations by integrations, and Kronecker δ's by Dirac δ-functions.

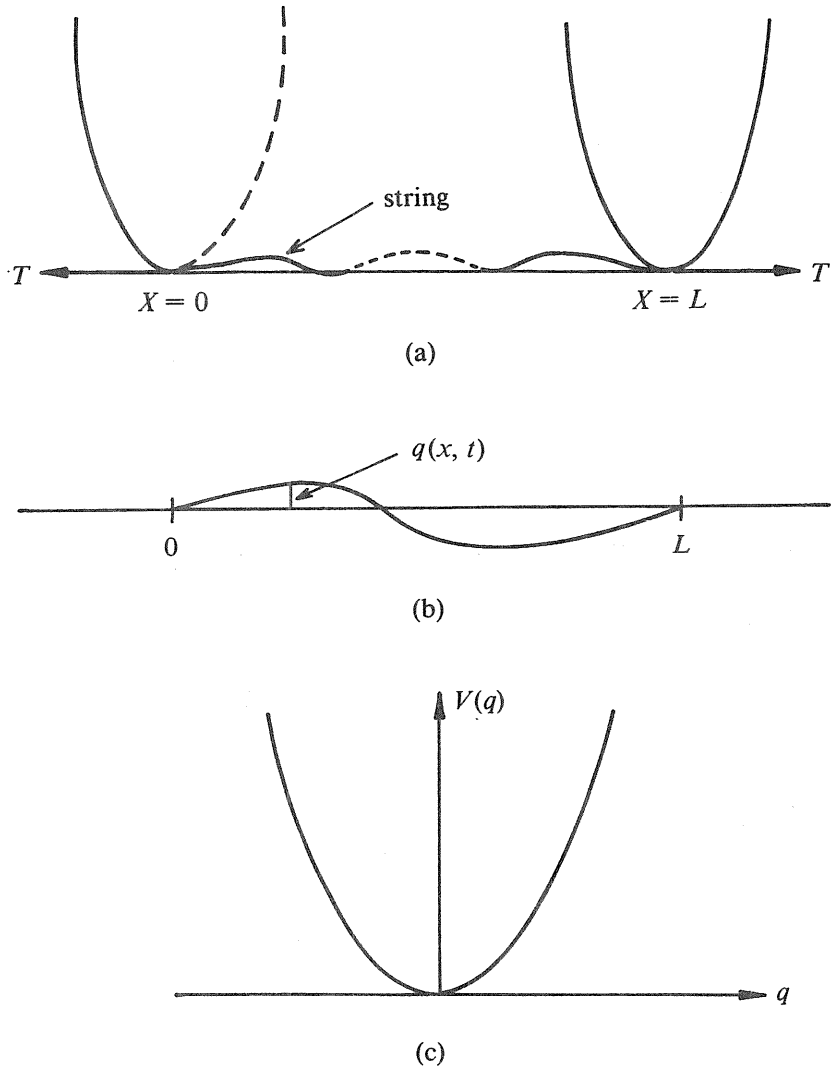


Fig. 1.1. (a) A string lies near the bottom of a cylinder. (b) $q(x, t)$ denotes the displacement of the string at (x, t) from its equilibrium position. (c) $V(q)$ describes the potential energy density as a function of q.

Just as in the discrete case, we may study the system in the Schrödinger representation. We need to replace the wave function $\psi(q, t)$ by a wave functional $\psi(q(x), t)$. In function $q(x)$ space, the momentum operator becomes a functional differentiation operator

$$\pi(x) \to \frac{1}{i} \frac{\delta}{\delta q(x)}. \tag{1.12}$$

The functional Schrödinger equation takes the form

$$\int_0^L dx \left[-\frac{1}{2\rho} \left(\frac{\delta}{\delta q(x)} \right)^2 + \frac{T}{2} \left(\frac{\partial q}{\partial x} \right)^2 + V(q) \right] \psi(q(x), t) = i \frac{\partial}{\partial t} \psi(q(x), t). \tag{1.13}$$

In Eq. (1.13), the first term is the generalization of Laplace operator to functional space. Obviously, this is not a simple equation to solve.

In the Heisenberg representation, we encounter t-dependent $q(x, t)$ and $\pi(x, t)$. The Heisenberg equations of motion are

$$\dot{q}(x, t) = \frac{1}{i} [q(x, t), H] = \pi(x, t)/\rho \tag{1.14}$$

$$\dot{\pi}(x, t) = \frac{1}{i} [\pi(x, t), H] . \tag{1.15}$$

Using the canonical quantization relations, we can reduce the second equation to

$$\begin{aligned} \dot{\pi}(x, t) &= \frac{1}{i} \left[\pi(x, t), \int_0^L dy \left(\frac{\pi(y, t)^2}{2\rho} + \frac{T}{2} \left(\frac{\partial q}{\partial y} \right)^2 + V(q(y)) \right) \right] \\ &= \frac{1}{i} \int_0^L dy \left\{ T \left(\frac{\partial q}{\partial y} \right) \left[\pi(x, t), \frac{\partial q(y, t)}{\partial y} \right] + [\pi(x, t), V(q(y))] \right\} \\ &= \frac{1}{i} \int dy \left\{ T \left(\frac{\partial q}{\partial y} \right) \left(-i \frac{\partial}{\partial y} \delta(x-y) \right) + V'(q(y)) (-i\delta(x-y)) \right\} \\ &= T \frac{\partial^2 q(x, t)}{\partial x^2} - V'(q(x, t)) . \end{aligned} \tag{1.16}$$

Except for the noncommutativity relations, these equations are the same as the classical equations of motion.

3. A Free Field

We consider the simple case in which $V(q)$ is quadratic in q. By proper scale transformations, we can absorb ρ and T into the new fields ϕ and $\dot{\phi}$, giving

$$\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{m^2}{2}\phi^2 \tag{1.17a}$$

$$\mathscr{H} = \frac{1}{2}\pi^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{m^2}{2}\phi^2 \tag{1.17b}$$

where ϕ and $\dot{\phi}$ obey canonical quantization relation. The Heisenberg equations of motion are

$$\pi(x, t) = \dot{\phi}(x, t) \tag{1.18}$$

$$\dot{\pi}(x, t) = \frac{1}{i}[\pi(x, t), H]$$

$$= \frac{\partial^2\phi}{\partial x^2} - m^2\phi(x, t). \tag{1.19}$$

Combining (1.18) and (1.19), we have

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + m^2\phi = 0. \tag{1.20}$$

Equation (1.20) is known as the Klein-Gordon equation. This equation together with equal-time commutation relations

$$[\phi, \phi] = [\pi, \pi] = 0$$

$$[\phi(x, t), \pi(y, t)] = i\,\delta(x - y)$$

determine the system completely.

Since (1.20) is linear in ϕ, it is easy to find its eigenfunctions. We begin with a finite string with the periodic boundary condition $\phi(0, t) = \phi(L, t)$. The eigenfunctions are of the form

$$e^{ikx - i\omega_k t} \quad \text{and} \quad e^{-ikx + i\omega_k t}. \tag{1.21}$$

The periodic boundary condition requires

$$kL = 2\pi n \quad \text{or} \quad k = 2\pi n/L \tag{1.22}$$

with

$$n = 0, \pm 1, \pm 2, \ldots .$$

Equation (1.20) determines the frequency ω_k:

$$-\omega_k^2 + k^2 + m^2 = 0 \tag{1.23}$$

or

$$\omega_k = \sqrt{k^2 + m^2}\,. \tag{1.24}$$

Note that we have taken both $e^{ikx - i\omega t}$ and its complex conjugate as eigenfunctions. Hence we only need to keep positive ω_k in (1.23). The notation $kx - \omega t$ is for future convenience. It will be identified as the relativistic scalar product.

We can expand $\phi(x, t)$ and $\dot{\phi}(x, t)$ in terms of the eigenfunctions

$$\phi(x, t) = \sum_k \frac{1}{\sqrt{2\omega_k L}} \left(a_k\, e^{ikx - i\omega_k t} + a_k^\dagger\, e^{-ikx + i\omega_k t}\right). \tag{1.25}$$

$$\pi(x, t) = \dot{\phi}(x, t)$$

$$= -i \sum_k \sqrt{\frac{\omega_k}{2L}} \left(a_k\, e^{ikx - i\omega_k t} - a_k^\dagger\, e^{-ikx + i\omega_k t}\right). \tag{1.26}$$

Their coefficients of expansion a_k and $a_k^\dagger$ are still operators, but are independent of x and t. The choice of normalization is to make a_k and $a_k^\dagger$ have simple physical interpretation.

To find the commutation relations between a_k and $a_k^\dagger$, we need to express a_k and $a_k^\dagger$ in terms of ϕ and π. We can accomplish this by taking fourier transforms. We note that both a_k and $a_k^\dagger$ appear in the fourier transforms. To obtain a_k alone, we need to fourier transform the combination $\omega_k \phi(x, t) + i\pi(x, t)$, giving

$$a_k = \frac{1}{\sqrt{2L\omega_k}} \int_0^L dx\, e^{-ikx + i\omega_k t} (\omega_k \phi(x, t) + i\pi(x, t)). \tag{1.27}$$

Similarly, we obtain

$$a_k^\dagger = \frac{1}{\sqrt{2L\omega_k}} \int_0^L dx\, e^{ikx - i\omega_k t} (\omega_k \phi(x, t) - i\pi(x, t)). \tag{1.28}$$

It is straightforward to verify that

$$[a_k, a_l] = [a_k^\dagger, a_l^\dagger] = 0. \tag{1.29}$$

The commutation between a_k and $a_l^\dagger$ gives

$$[a_k, a_l^\dagger] = \frac{1}{2L\sqrt{\omega_k \omega_l}} \int dxdy\, e^{-ikx + i\omega_k t}\, e^{ily - i\omega_l t}$$

$$\times [\omega_k \phi(x, t) + i\pi(x, t), \omega_l \phi(y, t) - i\pi(y, t)]$$

$$= \frac{1}{2L\sqrt{\omega_k \omega_l}} \int dxdy\, e^{-ikx + i\omega_k t}\, e^{ily - i\omega_l t}$$

$$\times (\omega_k \delta(x - y) + \omega_l \delta(x - y))$$

$$= \frac{\omega_k + \omega_l}{2L\sqrt{\omega_k \omega_l}} \int dx\, e^{-i(k-l)x + i(\omega_k - \omega_l)t}. \tag{1.30}$$

The x-integration leads to a Kronecker δ:

$$\int_0^L dx\, e^{-i(k-l)x} = L\delta_{kl}. \tag{1.31}$$

Under the condition $k = l$, the t-dependence drops out completely. We then have

$$[a_k, a_l^\dagger] = \delta_{kl}. \tag{1.32}$$

Equations (1.29) and (1.32) imply that a_k and $a_k^\dagger$ have the interpretation of annihilation and creation operators. The modes associated with different k's are decoupled.

Substituting (1.25) and (1.26) into the Hamiltonian (1.6) and (1.17), we have

$$H = \int dx\, \mathscr{H}$$

$$= \int_0^L dx \left[\frac{1}{2}\pi(x, t)^2 + \frac{1}{2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{2}m^2\phi^2\right]$$

$$= \int_0^L dx \left\{-\frac{1}{2}\sum_k \sum_l \frac{\sqrt{\omega_k \omega_l}}{2L}\left(a_k\, e^{ikx - i\omega_k t} - a_k^\dagger\, e^{-ikx + i\omega_k t}\right)\right.$$

$$\times \left(a_l\, e^{ilx-i\omega_l t} - a_l^\dagger\, e^{-ilx+i\omega_l t}\right)$$

$$+\frac{1}{2}\sum_k\sum_l \frac{1}{2L\sqrt{\omega_k\omega_l}}\left(ika_k\, e^{ikx-i\omega_k t} - ika_k^\dagger\, e^{-ikx+i\omega_k t}\right)$$

$$\times \left(ila_l\, e^{ilx-i\omega_l t} - ila_l^\dagger\, e^{-ilx+i\omega_l t}\right)$$

$$+\frac{1}{2}m^2\sum_k\sum_l \frac{1}{2L\sqrt{\omega_k\omega_l}}\left(a_k\, e^{ikx-i\omega_k t} + a_k^\dagger\, e^{-ikx+i\omega_k t}\right)$$

$$\times \left(a_l\, e^{ilx-i\omega_l t} + a_l^\dagger\, e^{-ilx-i\omega_l t}\right)\Bigg\}. \tag{1.33}$$

First, look at the $a_k\, a_l$ terms. The x-integrations always lead to the Kronecker delta, $\delta_{k+l,0}$. This Kronecker delta implies that $l = -k$ and $\omega_k = \omega_l$. The coefficient of $a_k a_l$ is proportional to

$$-\omega_k + \frac{k^2}{\omega_k} + \frac{m^2}{\omega_k} = 0. \tag{1.34}$$

Hence, the $a_k a_l$ terms cancel each other. This cancellation also occurs for $a_k^\dagger a_l^\dagger$ terms. For the $a_k^\dagger a_l$ and $a_k a_l^\dagger$ terms, instead of cancelling each other, these terms add. After some algebra, we have

$$H = \sum_k \omega_k \frac{a_k^\dagger\, a_k + a_k\, a_k^\dagger}{2}. \tag{1.35}$$

Making use of (1.32), we can write (1.35) as

$$H = \sum_k \omega_k \left(a_k^\dagger\, a_k + \tfrac{1}{2}\right). \tag{1.36}$$

The expression $a_k^\dagger a_k$ may be identified as the number operator for mode k.

The ground state of this system, $|0\rangle$, is annihilated by all a_k

$$a_k|0\rangle = 0, \qquad \text{all } k. \tag{1.37}$$

In the present system, the ground state is a nondegenerate state with the

lowest energy. It may be properly normalized to $\langle 0|0\rangle = 1$. We can describe an excited state by its occupation numbers $\{n_k\}$ in the k-modes:

$$|\{n_k\}\rangle = \prod_k \frac{(a_k^\dagger)^{n_k}}{\sqrt{n_k!}}|0\rangle . \tag{1.38}$$

The energy of this excited state is

$$E = \sum_k (n_k\omega_k + \omega_k/2) . \tag{1.39}$$

The ground state corresponds to $n_k = 0$ for all k. It is important to note that the ground state energy is not only nonvanishing, but actually diverges. The divergence of the zero-point energy is a general feature of a quantum field theory. Since we usually measure the energy of a system from and above its ground state energy, this zero-point energy does not have any observable effect. We can substract the unobserved infinite zero-point energy to give

$$E' \equiv E - E\,(\text{ground state}) = \sum_k n_k\omega_k . \tag{1.40}$$

For a state with finite occupation numbers, E' is finite.

We can also introduce the total momentum for the string. It is

$$\begin{aligned} P &= -\int_0^L dx\,\pi\frac{\partial\phi}{\partial x} \\ &= i\sum_k\sum_l \sqrt{\frac{\omega_k}{2L}}\sqrt{\frac{1}{2\omega_l L}}\int_0^L dx\left(a_k\,e^{ikx-i\omega_k t} - a_k^\dagger\,e^{-ikx+i\omega_k t}\right) \\ &\qquad \times\left(ila_l\,e^{ilx-i\omega_l t} - ila_l^\dagger\,e^{-ilx+i\omega_l t}\right). \end{aligned} \tag{1.41}$$

We look at the $a_k\,a_l$ terms. After x-integration, we find that only modes obeying $k + l = 0$ contribute. Then the summation over k and l vanishes identically,

$$\sum_k\sum_l l\,a_k\,a_l = \sum_k\sum_l \frac{1}{2}(k+l)\,a_k\,a_l = 0 .$$

Similarly, the $a_k^\dagger a_l^\dagger$ terms also cancel. The only remaining terms are $a_k a_l^\dagger$ and $a_k^\dagger a_l$ terms. After x-integration, we have

$$P = i \sum_k \sum_l \frac{1}{2L} \sqrt{\frac{\omega_k}{\omega_l}} (-ila_k a_l^\dagger - ila_k^\dagger a_l) L\delta_{kl}$$

$$= \sum_k k \frac{a_k^\dagger a_k + a_k a_k^\dagger}{2}$$

$$= \sum_k k a_k^\dagger a_k. \tag{1.42}$$

By using the symmetry of $k \leftrightarrow -k$, we have dropped the constant $\sum k/2$.

Thus for a state with occupation numbers $\{n_k\}$, the total momentum is

$$P = \sum n_k k. \tag{1.43}$$

We now concentrate on the energy and momentum associated with the excited modes. We identify them as the energy and momentum of the associated "quanta". We shall refer to these quanta as particles.

We call the ground state as the vacuum, the empty state. It contains no quanta. Its energy E' and momentum P are zero.

The state with $n_k = 1$ and all other occupation numbers being zeros is referred to as possessing a single quantum of momentum k and energy ω_k. We interpret this quantum as a particle with momentum k and energy

$$\omega_k = \sqrt{k^2 + m^2}. \tag{1.44}$$

The state with $n_k = n_l = 1$ and all other occupation numbers being zeros is a two-quantum state with momentum and energy (k, ω_k) and (l, ω_l) respectively. The total energy E' and momentum P of this 2-quantum state are

$$E' = \omega_k + \omega_l, \qquad P = k + l. \tag{1.45}$$

We may interpret this two-quantum state as a noninteracting two-particle state. In the present case, all multiparticle states are noninteracting.

The noninteracting field theory is a consequence of a quadratic potential $V(\phi) = m^2\phi^2/2$. If the potential $V(\phi)$ contains higher order terms such as a ϕ^4 term, then we have an interacting field theory. We shall learn how to handle the interacting case in the next chapter.

4. Nonrelativistic Many-Particle Systems

We have seen that a quantum field theory is intrinsically a many-particle system. It is capable of describing the interaction of an indefinite number of particles. On the other hand, the nonrelativistic quantum mechanics that we have studied earlier is a one-particle system. It describes the interaction of a single particle with a space-dependent potential $V(x)$. In the following, we shall use the quantum field theory language to describe a one-particle system. In a sense, we are introducing unnecessary complications into a simple system. As we shall see, we can gain some insight from this approach.

One way to turn a single particle theory into a field theory is to create many copies of the one-particle system. We also make an assumption that the particles so created do not interact among themselves. The one-particle system obeys the Schrödinger equation

$$i\frac{\partial\psi(x,t)}{\partial t} = \left[-\frac{1}{2m}\nabla^2 + V(x)\right]\psi(x,t). \tag{1.46}$$

For ease of explanation, we assume that all eigenstates are discrete, and that their wave functions $\psi(x, t)$ obey the normalization and completeness conditions

$$\int dx\, \psi^*_m(x,t)\, \psi_n(x,t) = \delta_{mn} \tag{1.47}$$

$$\sum_n \psi_n(x,t)\, \psi^*_n(y,t) = \delta(x-y). \tag{1.48}$$

We introduce the annihilation and the creation operators for the eigenmode n as a_n, $a_n^\dagger$ obeying the usual commutation relations,

$$[a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0,$$

$$[a_n, a_m^\dagger] = \delta_{nm}. \tag{1.49}$$

We can now introduce the field variables $\phi(x, t)$ and $\phi^*(x, t)$ as

$$\phi(x,t) = \sum_n \psi_n(x,t)\, a_n, \tag{1.50}$$

$$\phi^*(x,t) = \sum_n \psi^*_n(x,t)\, a_n^\dagger. \tag{1.51}$$

The field $\phi(x, t)$ so constructed obeys the linear (Schrödinger) equation

$$i\frac{\partial}{\partial t}\phi(x, t) = \left[-\frac{1}{2m}\nabla^2 + V(x)\right]\phi(x, t). \tag{1.52}$$

It is straightforward to establish the equal-time quantization relations among the field variables:

$$[\phi(x, t), \phi(y, t)] = [\phi^*(x, t), \phi^*(y, t)] = 0 \tag{1.53}$$

and

$$[\phi(x, t), \phi^*(y, t)] = \sum_n \psi_n(x, t)\,\psi_n^*(y, t) = \delta(x-y). \tag{1.54}$$

Equations (1.52)–(1.54) are the field theoretical description of the noninteracting one-particle system. We can derive these equations from a Lagrange function,

$$\mathscr{L} = i\phi^*(x, t)\,\dot{\phi}(x, t) - \frac{1}{2m}\nabla\phi^*\cdot\nabla\phi$$

$$- V(x)\,\phi^*(x, t)\,\phi(x, t). \tag{1.55}$$

It is easy to see that the momentum conjugate to $\phi(x, t)$ is

$$\pi(x, t) \equiv \frac{\partial\mathscr{L}}{\partial\dot{\phi}} = i\phi^*(x, t) \tag{1.56}$$

and the Hamiltonian density is

$$\mathscr{H} = \pi\dot{\phi} - \mathscr{L} = \frac{1}{2m}\nabla\phi^*\cdot\nabla\phi + V(x)\,\phi^*\phi. \tag{1.57}$$

In terms of ϕ and π, Eqs. (1.53) and (1.54) are indeed the usual quantization relations. The total Hamiltonian of the many-particle system is

$$H = \int dx\,\mathscr{H} = \sum_n \omega_n a_n^\dagger a_n \tag{1.58}$$

where ω_n is the one-particle energy of state n. Note that in (1.55) ϕ^* is

always on the left of ϕ. There is no zero-point energy in (1.58). The operators $\phi(x, t)$ and $\phi^*(x, t)$ may be interpreted as the annihilation and creation operators of a particle at x and t.

Just as in the string case, we can introduce the vacuum, $|0\rangle$, as the no-particle state. The 1-particle state, 2-particle state, . . . are

$$|1_n\rangle \equiv a_n^\dagger |0\rangle$$

$$|2_{m,n}\rangle \equiv a_m^\dagger a_n^\dagger |0\rangle, \ldots,$$

respectively. We can recover the one-particle Schrödinger wave function $\psi(x, t)$ from the field variable $\phi(x, t)$ and the one-particle state $|1\rangle$ via

$$\psi(x, t) = \langle 0|\phi(x, t)|1\rangle. \tag{1.59}$$

In addition, we can define a two-particle wave function as

$$\psi(x, y, t) \equiv \langle 0|\phi(x, t)\,\phi(y, t)\,|2\rangle = \psi(y, x, t). \tag{1.60}$$

From the canonical quantization rule, we find that the two-particle wave function is symmetric with respect to the interchange of x and y. Particles with this symmetry property are said to obey Bose-Einstein statistics. We wish to point out that we did not derive this property: we assumed it. In nature, there are particles which do not obey Bose-Einstein statistics. We shall discuss this point in more detail in the future.

In reality, particles do interact with each other. We consider a situation in which the particles not only interact with the external potential $V_1(x)$, but also interact with each other via a potential $V_2(x, y)$. Using the field theory language, we can describe the interaction easily as

$$H = \int dx \left[\frac{1}{2m}\nabla\phi^*(x, t)\cdot\nabla\phi(x, t) + V_1(x)\,\phi^*(x, t)\,\phi(x, t)\right]$$

$$+ \frac{1}{2}\int dxdy\,\phi^*(x, t)\,\phi(x, t)\,V_2(x, y)\,\phi^*(y, t)\,\phi(y, t). \tag{1.61}$$

In (1.61), $\phi^*(x, t)\phi(x, t)$ describes the particle density operator at position (x, t). The two-particle interaction energy is proportional to the product of particle densities at x and y. The factor 1/2 is introduced to take care of the double counting. This is one of the systems that we shall study in detail.

Chapter 2

RELATIVISTIC FREE FIELD THEORIES

1. Free Scalar Fields

A relativistic free particle obeys the energy-momentum relation,

$$E^2 = \mathbf{p}^2 + m^2. \tag{2.1}$$

This is the same energy-momentum relation as described for the string earlier. Thus, it is natural for us to describe free scalar particles by a Klein-Gordon equation

$$\left[\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2 + m^2\right]\phi = 0. \tag{2.2}$$

In a relativistic covariant language, we can write Eq. (2.2) as

$$\partial^2 \phi \equiv g_{\mu\nu}\partial_\mu\partial_\nu\phi = 0. \tag{2.3}$$

with diagonal $g_{\mu\nu} = (1, -1, -1, -1)$ and $g_{\mu\nu} = 0$ for $\mu \neq \nu$. The equal-time quantization relations are

$$[\phi(x, t), \phi(y, t)] = [\dot{\phi}(x, t), \dot{\phi}(y, t)] = 0 \tag{2.4a}$$

$$[\phi(x, t), \dot{\phi}(y, t)] = i\delta^3(x-y). \tag{2.4b}$$

We can reproduce everything that we did in the string case. In particular, we obtain the Lagrange function, Hamiltonian, and the momentum operator as

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2\end{aligned} \tag{2.5}$$

$$\mathrm{H} = \int d^3x\, \mathscr{H}$$

$$= \int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] \tag{2.6}$$

and

$$\mathbf{P} = -\int d^3x\, \dot{\phi}\nabla\phi. \tag{2.7}$$

Just as in the string case, we can expand ϕ and $\dot{\phi}$ in terms of travelling wave solutions and obtain the annihilation and the creation operators. If we put a system in a box of volume V and impose the periodic boundary condition, we obtain

$$\phi(x, t) = \sum_k \frac{1}{\sqrt{2\omega_k V}}(a_k\, e^{-ik\cdot x} + a_k^\dagger\, e^{-ik\cdot x}), \tag{2.8}$$

$$\pi(x, t) \equiv \dot{\phi}(x, t)$$

$$= -i\sum_k \sqrt{\frac{\omega_k}{2V}}\,(a_k\, e^{-ik\cdot x} - a_k^\dagger\, e^{-ik\cdot x}), \tag{2.9}$$

where

$$k\cdot x \equiv \omega_k t - \mathbf{k}\cdot\mathbf{x}$$

and a_k and $a_k^\dagger$ obey the commutation relations of annihilation and creation operators. We have similar expressions for H, $\mathbf{P}$ and the vacuum state $|0\rangle$ as

$$H = \sum_k \omega_k\,(a_k^\dagger a_k + 1/2) \tag{2.10}$$

$$\mathbf{P} = \sum_k \mathbf{k}\, a_k^\dagger a_k \tag{2.11}$$

$$a_k|0\rangle = \langle 0|\, a_k^\dagger = 0, \qquad \text{all } k\ . \tag{2.12}$$

The one-particle states are normalized according to Kronecker deltas

$$\langle k'|k\rangle = \langle 0|a_{k'}a_k^\dagger|0\rangle = \delta_{k'k}\,. \tag{2.13}$$

We wish to point out briefly how to take the infinite V limit. As $V \to \infty$, the eigenstates form a continuum, and should be normalized according to Dirac delta functions. The crucial difference of these two normalizations is that, in the box normalization, we encounter

$$\int d^3x\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = V\delta_{kk'}, \tag{2.14}$$

and, in the continuum normalization, we encounter

$$\int d^3x\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} = (2\pi)^3\delta^3(k-k'). \tag{2.15}$$

Thus, to change from the box normalization to the continuum normalization, we need to replace $V\delta_{kk'}$ by $(2\pi)^3\delta^3(k-k')$. We shall work out the first few cross sections in terms of the box normalization to demonstrate that the physical cross sections are independent of the box size V. After that, we shall use only the continuum normalization.

It is important to point out that in our theory, ϕ is real (hermitian)

$$\phi = \phi^*.$$

This property implies that the particles associated with ϕ are their own antiparticles, and hence, they must be neutral.

To describe charged particles, we need a complex field $\phi(x)$. Just as we can express every complex number by a pair of real numbers, we can express ϕ and ϕ^* by a pair of real fields ϕ_1, ϕ_2 via

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)) \tag{2.16}$$

$$\phi^*(x) = \frac{1}{\sqrt{2}}(\phi_1(x) - i\phi_2(x)). \tag{2.17}$$

The real fields ϕ_1 and ϕ_2 are also free fields if ϕ and ϕ^* are.

Since we know the Lagrange density $\mathscr{L}$ and the Hamiltonian density $\mathscr{H}$ in terms of the real fields ϕ_1 and ϕ_2, we can translate them in terms of ϕ and ϕ^* as

$$\begin{aligned}\mathscr{L} &= \sum_{i=1}^{2}\left[\frac{1}{2}(\partial_\mu\phi_i)^2 - \frac{m^2}{2}\phi_i^2\right] \\ &= \partial_\mu\phi^*\,\partial_\mu\phi - m^2\phi^*\phi,\end{aligned} \tag{2.18}$$

$$\mathscr{H} = \sum_{i=1}^{2} \left[\frac{1}{2} |\dot{\phi}_i|^2 + \frac{1}{2} |\nabla \phi_i|^2 + \frac{m^2}{2} \phi_i^2 \right]$$

$$= \dot{\phi}^* \dot{\phi} + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi. \tag{2.19}$$

Similarly, the commutation relations

$$[\phi_i(x, t), \phi_j(y, t)] = [\dot{\phi}_i(x, t), \dot{\phi}_j(y, t)] = 0 \tag{2.20}$$

$$[\phi_i(x, t), \dot{\phi}_j(y, t)] = i\delta_{ij} \delta^3(x-y) \tag{2.21}$$

imply the equal-time quantization rules

$$[\phi, \phi] = [\phi, \phi^*] = [\phi^*, \phi^*] = 0$$

$$[\dot{\phi}, \dot{\phi}] = [\dot{\phi}, \dot{\phi}^*] = [\dot{\phi}^*, \dot{\phi}^*] = 0 \tag{2.22}$$

$$[\phi, \dot{\phi}] = [\phi^*, \dot{\phi}^*] = 0 \tag{2.23}$$

$$[\phi(x, t), \dot{\phi}^*(y, t)] = [\phi^*(x, t), \dot{\phi}(y, t)] = i\delta^3(x-y). \tag{2.24}$$

Just as in the real scalar case, we can expand ϕ and ϕ^* as combinations of creation and annihilation operators. In the continuum limit, we have

$$\phi(x) = \int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} (e^{-ik\cdot x} b_k + e^{ik\cdot x} c_k^\dagger) \tag{2.25}$$

$$\phi^*(x) = \int \frac{d^3k}{\sqrt{2\omega_k (2\pi)^3}} (e^{-ik\cdot x} c_k + e^{ik\cdot x} b_k^\dagger) \tag{2.26}$$

where

$$b_k = \frac{(a_1)_k + i(a_2)_k}{\sqrt{2}} \tag{2.27}$$

$$c_k = \frac{(a_1)_k - i(a_2)_k}{\sqrt{2}}. \tag{2.28}$$

Equation (2.25) tells us that ϕ annihilates a particle or creates an antiparticle. As free fields, the theory is symmetric under the interchange

of particle and antiparticle. Particles and their antiparticles can now have nonvanishing charges. The symmetry that we have just mentioned is a special case of the charge conjugation symmetry.

2. Free Dirac Equations

A Dirac particle, such as an electron, obeys a 4-component wave equation known as the Dirac equation,

$$(i\partial\!\!\!/ - m)\,\psi \equiv (i\gamma_\mu \partial_\mu - m)\,\psi = 0. \tag{2.29}$$

In (2.29), γ_μ are 4×4 matrices, known as Dirac gamma matrices. They obey the anti-commutation relations

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}\mathbf{1}, \quad \mathbf{1} = \text{identity matrix.} \tag{2.30}$$

A well-known set of gamma matrices is

$$\gamma^0 = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.31a}$$

$$\gamma^i = -\gamma_i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{2.31b}$$

where σ^i are the usual Pauli matrices. This set of gamma matrices is known as the standard representation.

We shall give a short review of gamma matrices and some transformation properties associated with Dirac equations.

(1) The products of gamma matrices form 16 linearly independent 4×4 matrices. They are

$$\begin{array}{ll} \mathbf{1} & 1 \\ \gamma_\mu & 4 \\ \sigma_{\mu\nu} \equiv \dfrac{i}{2}[\gamma_\mu, \gamma_\nu] & 6 \\ \gamma_\mu\gamma_5 & 4 \\ \gamma_5 \equiv i\,\gamma^0\gamma^1\gamma^2\gamma^3 & 1 \\ & \overline{16} \end{array} \tag{2.32}$$

In particular, $\gamma_5^2 = 1$ and γ_5 anti-commutes with γ_μ.

(2) These matrices obey the matrix commutation relations

$$\left[\frac{1}{2}\sigma_{\mu\nu}, \mathbf{1}\right] = \left[\frac{1}{2}\sigma_{\mu\nu}, \gamma_5\right] = 0 \tag{2.33}$$

$$\frac{1}{i}\left[\frac{1}{2}\sigma_{\mu\nu}, \gamma_\lambda\right] = g_{\nu\lambda}\gamma_\mu - g_{\mu\lambda}\gamma_\nu \tag{2.34}$$

$$\frac{1}{i}\left[\frac{1}{2}\sigma_{\mu\nu}, \gamma_\lambda\gamma_5\right] = g_{\nu\lambda}\gamma_\mu\gamma_5 - g_{\mu\lambda}\gamma_\nu\gamma_5 \tag{2.35}$$

$$\begin{aligned}\frac{1}{i}\left[\frac{1}{2}\sigma_{\mu\nu}, \frac{1}{2}\sigma_{\lambda\rho}\right] &= g_{\nu\lambda}\frac{1}{2}\sigma_{\mu\rho} - g_{\mu\lambda}\frac{1}{2}\sigma_{\nu\rho} \\ &\quad + g_{\nu\rho}\frac{1}{2}\sigma_{\lambda\mu} - g_{\mu\rho}\frac{1}{2}\sigma_{\lambda\nu}.\end{aligned} \tag{2.36}$$

To understand the meaning of these equations, we look at (2.36) and obtain

$$\frac{1}{i}\left[\frac{1}{2}\sigma_{23}, \frac{1}{2}\sigma_{31}\right] = \frac{1}{2}\sigma_{12}.$$

If we identify σ_{12} as σ_3, etc, we have

$$\frac{1}{i}\left[\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2\right] = \frac{1}{2}\sigma_3. \tag{2.37}$$

This is, of course, one of the commutation relations among the generators of the 3-dimensional rotation group. Equation (2.36) tells us that $\frac{1}{2}\sigma_{\mu\nu}$ transforms as the generators of a homogeneous Lorentz group. Equations (2.33)–(2.35) imply that $\mathbf{1}$ and γ_5 transform as scalars, and γ_μ and $\gamma_\mu\gamma_5$ transform as 4-vectors under this Lorentz group. These relations are crucial for the verification of Lorentz invariance of a Dirac system.

We shall work out one of the equations. The general techniques work for the others. Note that

$$\frac{1}{i}\frac{1}{2}\sigma_{\mu\nu} = \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$
$$= \frac{1}{2}\gamma_\mu\gamma_\nu - \frac{1}{2}g_{\mu\nu}\mathbf{1}. \tag{2.38}$$

The $g_{\mu\nu}$ term obviously commutes with every matrix. We only need to consider the $\gamma_\mu\gamma_\nu$ term. Thus

$$\frac{1}{i}\left[\frac{1}{2}\sigma_{\mu\nu}, \gamma_\lambda\right] = \frac{1}{2}[\gamma_\mu\gamma_\nu, \gamma_\lambda]$$

$$= \frac{1}{2}(\gamma_\mu\gamma_\nu\gamma_\lambda - \gamma_\lambda\gamma_\mu\gamma_\nu)$$

$$= \frac{1}{2}(\gamma_\mu\{\gamma_\nu, \gamma_\lambda\} - \{\gamma_\lambda, \gamma_\mu\}\gamma_\nu)$$

$$= g_{\nu\lambda}\gamma_\mu - g_{\lambda\mu}\gamma_\nu$$

as desired.

(3) By Lorentz invariance, we mean that for every solution $\psi(x)$ to a Dirac equation, and for every Lorentz transformation

$$x'^\mu = a^\mu{}_\nu x^\nu, \tag{2.39}$$

there exists a transformed solution in the new coordinates x'. This transformed solution ψ' is linearly related to the original solution

$$\psi'(x') = S(a)\,\psi(x) \tag{2.40}$$

such that the 4×4 transformation matrix forms a representation of the Lorentz group. In other words, if

$$x' = a_2 x$$

$$x'' = a_1 x'$$

are two Lorentz transformations, and

$$\psi'(x') = S(a_2)\,\psi(x)$$

$$\psi''(x'') = S(a_1)\,\psi'(x')$$

are the corresponding transformations on ψ, then the combined transformation a_1a_2 requires

$$\psi''(x'') = S(a_1)\,S(a_2)\,\psi(x)$$

$$= S(a_1a_2)\,\psi(x)$$

or

$$S(a_1a_2) = S(a_1)\,S(a_2). \tag{2.41}$$

Thus, the consistency condition requires that $S(a)$ transforms as a representation of the Lorentz group.

Under a pure rotation in 3-space, we know

$$S(a) = e^{i\omega\cdot\sigma/2}$$

where ω is the rotation angle. The corresponding representation $S(a)$ under a homogeneous Lorentz transformation is

$$S(a) = \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}\frac{\sigma_{\mu\nu}}{2}\right),\ \omega_{\mu\nu} = -\,\omega_{\nu\mu}\,. \tag{2.42}$$

We call ψ obeying (2.40) a Dirac spinor. The transformation law for γ^μ is

$$S^{-1}(a)\,\gamma^\mu S(a) = a^\mu{}_\nu\gamma^\nu. \tag{2.43}$$

Equation (2.43) also provides us with the relations between $a^\mu{}_\nu$ and the generalized angle $\omega_{\mu\nu}$.

It is easy to verify that if ψ obeys the Dirac equation and $S(a)$ obeys (2.43), then $\psi'(x')$ obeys the Dirac equation in the new coordinates x'. Consider the expression

$$S^{-1}(a)\,(i\gamma^\mu\partial'_\mu - m)\,\psi'(x')$$

$$= S^{-1}(a)\,(i\gamma^\mu\partial'_\mu - m)\,S(a)\,\psi(x)$$

$$= (iS^{-1}\,\gamma^\mu S\,\partial'_\mu - m)\,\psi(x) \tag{2.44}$$

Since both $S^{-1}\gamma^{\mu}S$ and ∂'_{μ} transform as 4-vectors

$$S^{-1}\gamma^{\mu}S = a^{\mu}{}_{\nu}\gamma^{\nu} \tag{2.45a}$$

$$\partial'_{\mu} = a_{\mu}{}^{\lambda}\partial_{\lambda} = -a^{\lambda}{}_{\mu}\partial_{\lambda}, \tag{2.45b}$$

we have

$$\begin{aligned} & S^{-1}(i\gamma^{\mu}\partial'_{\mu} - m)\,\psi'(x') \\ & = (i\,a^{\mu}{}_{\nu}\,a_{\mu}{}^{\lambda}\,\gamma^{\nu}\,\partial_{\lambda} - m)\,\psi(x) \\ & = (i\delta^{\lambda}_{\nu}\,\gamma^{\nu}\,\partial_{\lambda} - m)\,\psi(x) = (i\partial\!\!\!/ - m)\,\psi(x). \end{aligned} \tag{2.46}$$

Thus, if $\psi(x)$ obeys the Dirac equation, so does $\psi'(x')$, as desired.

(4) From the transformation law of ψ, we can obtain the transformation law of $\psi^{\dagger}$,

$$\begin{aligned} \psi'^{\dagger}(x') &= (S(a)\,\psi(x))^{\dagger} \\ &= \psi^{\dagger}(x)\exp\left(\frac{i}{4}\,\omega_{\mu\nu}\sigma^{\dagger}_{\mu\nu}\right) \\ &\neq \psi^{\dagger}(x)\,S^{-1}(a). \end{aligned} \tag{2.47}$$

Thus, $\psi^{\dagger}$ does not transform covariantly. To obtain a covariant transformation law, we need to convert $\sigma^{\dagger}_{\mu\nu}$ to $\sigma_{\mu\nu}$. Note that σ_{ij} is hermitian and σ_{0i} is antihermitian, we then have

$$\gamma^0\sigma^{\dagger}_{ij}\gamma^0 = \gamma^0\sigma_{ij}\gamma^0 = \sigma_{ij} \tag{2.48}$$

and

$$\gamma^0\sigma^{\dagger}_{0i}\gamma^0 = \gamma^0(-\sigma_{0i})\,\gamma^0 = \sigma_{0i}. \tag{2.49}$$

Then, we have for all μ, ν,

$$\gamma^0\sigma^{\dagger}_{\mu\nu}\gamma^0 = \sigma_{\mu\nu}. \tag{2.50}$$

This implies that $\bar{\psi} \equiv \psi^\dagger \gamma^0$ transforms covariantly

$$\begin{aligned}
\bar{\psi}'(x') &\equiv \psi^{\dagger'}(x')\,\gamma^0 \\
&= \psi^\dagger(x)\, e^{-\frac{i}{4}\omega_{\mu\nu}\sigma_{\mu\nu}^\dagger}\gamma^0 \\
&= \bar{\psi}(x)\,\gamma^0\, e^{-\frac{i}{4}\omega_{\mu\nu}\sigma_{\mu\nu}^\dagger}\gamma^0 \\
&= \bar{\psi}(x)\, e^{-\frac{i}{4}\omega_{\mu\nu}\gamma^0\sigma_{\mu\nu}^\dagger\gamma^0} \\
&= \bar{\psi}(x)\, e^{-\frac{i}{4}\omega_{\mu\nu}\sigma_{\mu\nu}} = \bar{\psi}(x)\, S^{-1}(a). \qquad (2.51)
\end{aligned}$$

We call $\bar{\psi}$ the Dirac adjoint of ψ.

It is straightforward to show that

$\bar{\psi}\,\psi$ and $\bar{\psi}\,\gamma_5\psi$ are scalars,
$\bar{\psi}\,\gamma_\mu\psi$ and $\bar{\psi}\,\gamma_\mu\gamma_5\psi$ are vectors,

and

$\bar{\psi}\sigma_{\mu\nu}\psi$ is an antisymmetric tensor.

Let us verify the transformation law of $\bar{\psi}\gamma^\mu\psi$. Using (2.40), (2.45a) and (2.51), we have

$$\begin{aligned}
\bar{\psi}'(x')\,\gamma^\mu\,\psi'(x') &= \bar{\psi}(x)\,S^{-1}(a)\,\gamma^\mu\,S(a)\,\psi(x) \\
&= a^\mu{}_\nu\,\bar{\psi}(x)\,\gamma^\nu\,\psi(x).
\end{aligned}$$

The transformation laws of other bilinear expressions may be worked out in a similar way.

(5) Space inversion

The space inversion is a discrete Lorentz transformation

$$x'^\mu = (x^0, -\mathbf{x}) = a^\mu{}_\nu x^\nu. \qquad (2.52)$$

The corresponding $S(a)$ is

$$S(a) = \gamma^0 \tag{2.53}$$

satisfying

$$S^{-1}\gamma^\mu S = (\gamma^0, -\boldsymbol{\gamma}) = a^\mu{}_\nu x^\nu. \tag{2.54}$$

Hence, the transformed Dirac wave function under space inversion is

$$\psi'(x') = \gamma^0 \psi(x). \tag{2.55}$$

Using (2.55), we can show that $\bar{\psi}\psi$ is a true scalar, $\bar{\psi}\gamma_5\psi$ is a pseudoscalar, $\bar{\psi}\gamma_5\psi$ is a polar 4-vector, and $\bar{\psi}\gamma_\mu\gamma_5\psi$ is an axial vector.

(6) Majorana representation and charge conjugation solution

It is possible to construct a representation of gamma matrices such that

γ^0 is imaginary and antisymmetric,
$\boldsymbol{\gamma}$ is imaginary and symmetric.

We call such a representation a Majorana representation. In a Majorana representation, the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x, t) = 0$$

is a real equation. Hence, if ψ is a solution, so is

$$\psi_c \equiv \psi^* . \tag{2.56}$$

Spinor ψ_c is called the charge conjugation solution of ψ.

A spinor ψ obeying the reality condition

$$\psi = \psi^* \tag{2.57}$$

is called a Majorana spinor. A Dirac spinor is made of 2 majorana spinors, just as a charged scalar field is made of two real scalar fields.

3. Plane Wave Solutions to Dirac Equations

For each momentum p, we have two positive energy solutions and two negative energy solutions. The positive energy E is given by

$$E = \sqrt{\mathbf{p}^2 + m^2}.$$

In the standard representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and in the rest frame ($\mathbf{p} = 0$), we have

$$\left(i\gamma^0 \frac{\partial}{\partial t} - m \right) \psi = 0. \tag{2.58}$$

This gives rise to four independent solutions

$$\begin{aligned} \psi_1 &= e^{-imt} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \psi_2 = e^{-imt} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ \psi_3 &= e^{imt} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \psi_4 = e^{imt} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{2.59}$$

The first two solutions have positive energy and the next two solutions have negative energy. The solutions for $\mathbf{p} \neq 0$ can be written as

$$\psi_+ = e^{i\mathbf{p}\cdot\mathbf{x} - iEt} u(p, s), \quad s = \pm 1 \tag{2.60a}$$

$$\psi_- = e^{-i\mathbf{p}\cdot\mathbf{x} + iEt} v(p, s), \quad s = \pm 1 . \tag{2.60b}$$

In (2.60), phases $\pm(i\mathbf{p}\cdot\mathbf{x}\text{-}iEt)$ are the Lorentz invariant expressions of $\mp imt$. The spinor u and v are $S(a)$ times the upper and the lower spinors in Eq. (2.59). At $p = 0$, $\psi_\pm$ reduces to expression (2.59).

The Dirac equation implies that

$$\begin{aligned} (i\partial\!\!\!/ - m)\, \psi_+ &= [\, i\gamma^\mu (-ip_\mu) - m \,]\, e^{-ip\cdot x} u(p, s) \\ &= e^{-ip\cdot x} (p\!\!\!/ - m)\, u(p, s) = 0 . \end{aligned} \tag{2.61}$$

Hence, we obtain

$$(\not{p} - m)\, u(p, s) = 0. \tag{2.62}$$

Similarly, we have

$$(i\not{\partial} - m)\, \psi_- = -e^{ip\cdot x}(\not{p} + m)\, v(p, s) = 0. \tag{2.63}$$

Taking Dirac conjugation, we also have

$$\bar{u}(p, s)\,(\not{p} - m) = 0, \tag{2.64}$$

$$\bar{v}(p, s)\,(\not{p} + m) = 0. \tag{2.65}$$

We may interpret Eqs. (2.62) and (2.63) as projections into the positive and the negative solution subspaces. We can introduce projection operators $\Lambda_\pm$ by

$$\Lambda_\pm = \frac{m \pm \not{p}}{2m}. \tag{2.66}$$

For particles on the mass shell, $p^2 = m^2$, we have

$$\Lambda_+^2 = \frac{(\not{p} + m)^2}{4m^2} = \frac{p^2 + 2m\not{p} + m^2}{4m^2} = \Lambda_+ \tag{2.67a}$$

$$\Lambda_-^2 = \frac{(m - \not{p})^2}{4m^2} = \frac{m^2 - 2m\not{p} + p^2}{4m^2} = \Lambda_- \tag{2.67b}$$

$$\Lambda_+\Lambda_- = \Lambda_-\Lambda_+ = 0. \tag{2.67c}$$

Thus, Eqs. (2.62) and (2.63) tell us that

$$\Lambda_+ u = u, \qquad \Lambda_- u = 0 \tag{2.68a}$$

$$\Lambda_+ v = 0, \qquad \Lambda_- v = v. \tag{2.68b}$$

Next, we wish to normalize the spinors, u and v's. Note that $\bar{\psi}\psi$ is a scalar, and $\psi^\dagger\psi$ is not! Hence, we choose the normalization among u and v as

$$\bar{u}(p, s)\, u(p, s') = \delta_{ss'}, \tag{2.69a}$$

$$\bar{u}(p, s)\, v(p, s') = \bar{v}(p, s)\, u(p, s') = 0 \tag{2.69b}$$

$$\bar{v}(p, s)\, v(p, s') = -\delta_{ss'}. \tag{2.69c}$$

The first two equations give no surprise. The negative sign in the last expression may be bothersome. However, we can understand the sign by going to the rest frame,

$$\bar{v}(p, s)\, v(p, s) = (0\ v^{\dagger}(s)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ v(s') \end{pmatrix}$$

$$= -v^{\dagger}(s)\, v(s') = -\delta_{ss'}.$$

We can not change the sign of u and v by any Lorentz transformation. This sign also enters the completeness condition

$$\sum_s u(p, s)\, \bar{u}(p, s) - \sum_s v(p, s)\, \bar{v}(p, s) = 1. \tag{2.70}$$

Multiplying (2.70) by $\Lambda_{\pm}$ we have

$$\sum_s u(p, s)\, \bar{u}(p, s) = \Lambda_+ = \frac{\not{p} + m}{2m}, \tag{2.71}$$

$$-\sum_s v(p, s)\, \bar{v}(p, s) = \Lambda_- = \frac{m - \not{p}}{2m}. \tag{2.72}$$

These relations are important for future applications.

4. Free Dirac Fields

A free Dirac field ψ obeys

$$(i\not{\partial} - m)\psi = 0 \tag{2.73}$$

and may be derived from the Lagrange function

$$\mathscr{L} = \bar{\psi}(i\not{\partial} - m)\, \psi. \tag{2.74}$$

The momentum variable conjugate to ψ is

$$\pi = \frac{\partial \mathscr{L}}{\partial \dot{\psi}} = i\psi^{\dagger}. \tag{2.75}$$

We simply state that it is an empirical fact that many-particle Dirac wave functions are totally antisymmetric. Thus, the canonical quantization rules are given by the equal-time anticommutation relations

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0 \tag{2.76}$$

and

$$\{\psi_\alpha(x, t), \psi_\beta^\dagger(y, t)\} = \delta_{\alpha\beta}\,\delta^3(x - y). \tag{2.77}$$

In (2.76) and (2.77), the curl brackets describe anticommutators.

Just as in the scalar case, we can expand the field variables ψ and $\bar{\psi}$ as linear combinations of plane wave solutions, obtaining

$$\psi(x, t) = \sum_s \int d^3p \sqrt{\frac{m}{(2\pi)^3 E_p}} [b(p, s)\, u(p, s)\, e^{-ip\cdot x} + d^\dagger(p, s)\, v(p, s)\, e^{ip\cdot x}] \tag{2.78}$$

$$\bar{\psi}(x, t) = \sum_s \int d^3p \sqrt{\frac{m}{(2\pi)^3 E_p}} [b^\dagger(p, s)\, \bar{u}(p, s)\, e^{ip\cdot x} + d(p, s)\, \bar{v}(p, s)\, e^{-ip\cdot x}] \tag{2.79}$$

with

$$E_p = \sqrt{\mathbf{p}^2 + m^2}. \tag{2.80}$$

Coefficients b, d, $b^\dagger$, $d^\dagger$ are time and space independent operators. From the canonical quantization relations (2.76), (2.77), we can work out the anticommutation relations among b, d, $b^\dagger$, $d^\dagger$, giving

$$\{b, b\} = \{d, d\} = \{b, d\} = 0 \tag{2.81}$$

$$\{b^\dagger, b^\dagger\} = \{d^\dagger, d^\dagger\} = \{b^\dagger, d^\dagger\} = 0. \tag{2.82}$$

$$\{b(p, s), b^\dagger(p', s')\} = \{d(p, s), d^\dagger(p', s')\} = \delta_{ss'}\delta^3(p - p') \tag{2.83}$$

$$\{b, d^\dagger\} = \{d, b^\dagger\} = 0. \tag{2.84}$$

Just as in the case of the charged scalar field, we interpret b and d as the annihilation operators of the particle and of its antiparticle respectively. We interpret $b^\dagger$ and $d^\dagger$ as the creation operators of the particle and of its antiparticle respectively. In terms of Dirac field operators, we find that ψ can either annihilate a particle or create an antiparticle, while $\bar{\psi}$ (or $\psi^\dagger$) can either create a particle or annihilate an antiparticle.

We also define the vacuum state $|0\rangle$ as a state annihilated by all annihilation operators

$$b(p, s)|0\rangle = d(p, s)|0\rangle = 0. \tag{2.85}$$

The one-particle and one-antiparticle state are

$$b^\dagger(p, s)|0\rangle \quad \text{and} \quad d^\dagger(p, s)|0\rangle,$$

respectively. We can introduce many particle and antiparticle states straightforwardly.

We need to justify that the vacuum state so defined is indeed the lowest energy state of the system. The energy operator is the Hamiltonian of the system,

$$\begin{aligned} H &= \int d^3x\, \mathscr{H} \\ &= \int d^3x \left[\frac{\partial \mathscr{L}}{\partial \dot{\psi}} \dot{\psi} - \mathscr{L}\right] \\ &= \int d^3x\, \bar{\psi}(i\boldsymbol{\gamma}\cdot\nabla + m)\,\psi. \end{aligned} \tag{2.86}$$

After substituting (2.78) and (2.79) into (2.86) and carrying out the x-integration, we have

$$\begin{aligned} H &= \sum_s \int d^3p\, E_p [b^\dagger(p, s)\, b(p, s) - d(p, s)\, d^\dagger(p, s)] \\ &= \sum_s \int d^3p\, [E_p (b^\dagger(p, s)\, b(p, s) + d^\dagger(p, s)\, d(p, s)) \\ &\quad - E_p \delta^3(p - p)\, \delta_{ss}]. \end{aligned} \tag{2.87}$$

The last term represents an infinite negative energy. If we ignore this infinite negative constant, we arrive at

$$H' = \sum_s \int d^3p\, E_p (b^\dagger(p, s)\, b(p, s) + d^\dagger(p, s)\, d(p, s)). \tag{2.88}$$

H' is obviously a nonnegative operator. Let $\{n^\pm(p, s)\}$ be the particle and antiparticle occupation numbers of a state. The energy of this state is

$$E' = \sum_s \int d^3p\, E_p \{n^+(p, s) + n^-(p, s)) \geq 0. \tag{2.89}$$

In a similar manner, we can introduce the momentum operator as

$$\mathbf{P} = \int d^3x\, \bar{\psi}(-i\nabla\psi)\,. \tag{2.90}$$

In terms of the creation and annihilation operators, we can express $\mathbf{P}$ as

$$\mathbf{P} = \sum_s \int d^3p\, \mathbf{p}\, (n^+(p, s) + n^-(p, s)). \tag{2.91}$$

In deriving (2.91), we have ignored a summation over constant vectors via symmetry argument as before.

5. Negative Energy Sea

We wish to give a brief physical explanation of the infinite negative constant appearing in the Dirac ground state. If we follow our construction of a nonrelativistic field variable closely, we would construct a Dirac field as

$$\psi(x, t) = \sum_s \int d^3p \sqrt{\frac{m}{(2\pi)^3 E_p}} \left[a_+(\mathbf{p}, s)\, u(\mathbf{p}, s)\, e^{i\mathbf{p}\cdot\mathbf{x} - iE_p t}\right.$$

$$\left. + a_-(\mathbf{p}, s)\, v(-\mathbf{p}, -s)\, e^{i\mathbf{p}\cdot\mathbf{x} + iE_p t}\right]. \tag{2.92}$$

In (2.92), $a_+(p, s)$ is the annihilation operator of a positive energy particle, and $a_-(p, s)$ is the annihilation operator of a negative energy particle. Similarly, we can express $\bar{\psi}(x, t)$ as a combination of their respective creation operators $a_+^\dagger(p, s)$ and $a_-^\dagger(p, s)$. The Hamiltonian operator is

$$H = \sum_s \int d^3p \, [E_p \, a_+^\dagger \, (p, s) \, a_+ \, (p, s)$$

$$+ (-E_p) \, a_-^\dagger \, (p, s) \, a_- \, (p, s)]. \tag{2.93}$$

The $a_\pm$ and $a_\pm^\dagger$ obey the anticommutation relations as expected. In the present form, there is no additive constant because ψ is always on the right of $\bar{\psi}$. We can introduce a different vacuum state, $|\,\rangle'$, which is annihilated by both a_+ and a_-,

$$a_+ \, (p, s) \, |\,\rangle' = a_- \, (p, s) \, |\,\rangle' = 0. \tag{2.94}$$

This vacuum state contains no particles of either positive or negative energy.

The trouble of this vacuum state $|\,\rangle'$ is that it is not the lowest energy state. Obviously, a negative energy particle state has a lower energy than that of the vacuum state $|\,\rangle'$. Dirac proposed a solution to this difficulty when he introduced his equation to describe electrons. For the ease of exposition, we shall refer to our Dirac particles as electrons.

Dirac proposed that, in the physical vacuum $|0\rangle$, all negative energy states are occupied and all positive energy electron states are empty. (See Fig. 2.1). Since electrons are fermions obeying the exclusion principle, we cannot create any additional negative energy electrons in the vacuum. Thus, we can describe the ground state by

$$a_+ \, (p, s) \, |0\rangle = a_-^\dagger \, (p, s) \, |0\rangle = 0 \, . \tag{2.95}$$

The positive energy one-electron state is $a_+^\dagger \, (\mathbf{p}, s) \, |0\rangle$. The state with a negative energy electron removed is called a "hole", or a positron. It is given by $a_- \, (-\mathbf{p}, -s) \, |0\rangle$. The positron state described above has a

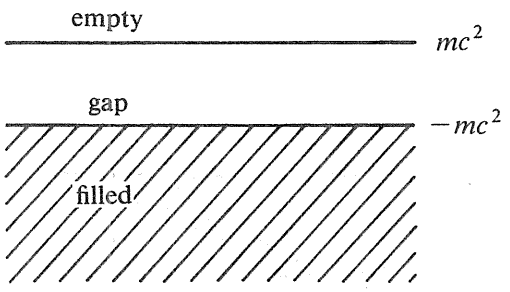


Fig. 2.1 In the physical vacuum, all negative energy states are occupied and all positive energy states are empty.

positive energy E, momentum $\mathbf{p}$, and spin s. Figure (2.2) describes the creation of positive energy electron and a negative energy hole, in the form of an electron-positron pair.

The crucial observation is that by filling up the undesirable negative energy states, we effectively interchange the roles of creation and annihilation operators for these modes. Indeed, if we make the identification

$$a_+(\mathbf{p}, s) \longleftrightarrow b(\mathbf{p}, s) \tag{2.96}$$

$$a_-^\dagger(-\mathbf{p}, -s) \longleftrightarrow d(\mathbf{p}, s) \tag{2.97}$$

we recover our result in the previous section. The canonical anticommutation relations are not affected by the interchange of creation and annihilation operators. We can now interpret the negative infinite constant in H as the energy associated with all these unobserved electrons in the negative energy sea. The positrons are nothing but the empty bubbles in the mid of the negative energy ocean.

You may feel uneasy about the ghostly appearance played by the positrons. Can we describe the same system by the presence of positive energy positrons, and interpret the electrons as holes in the negative energy positron sea? The answer is yes. In Fig. (2.3a), we describe a naive electron vacuum. Fig. (2.3b) represents the filled negative electron sea. Since positrons represent "empty spaces", the naive positron vacuum in the electron description is shown in Fig. (2.3c). A negative energy positron state is an empty bubble in the upper (positive energy electron) region of the positron vacuum. For Mr. Dirac, who lived in a positronic world, he would suggest filling all the negative energy positron sea which

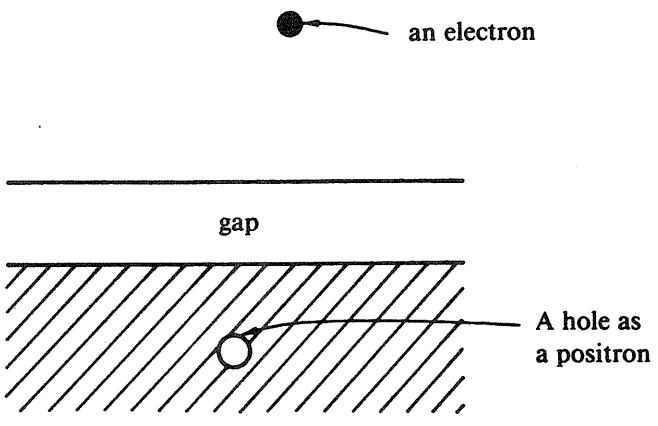


Fig. 2.2 An electron-positron pair.

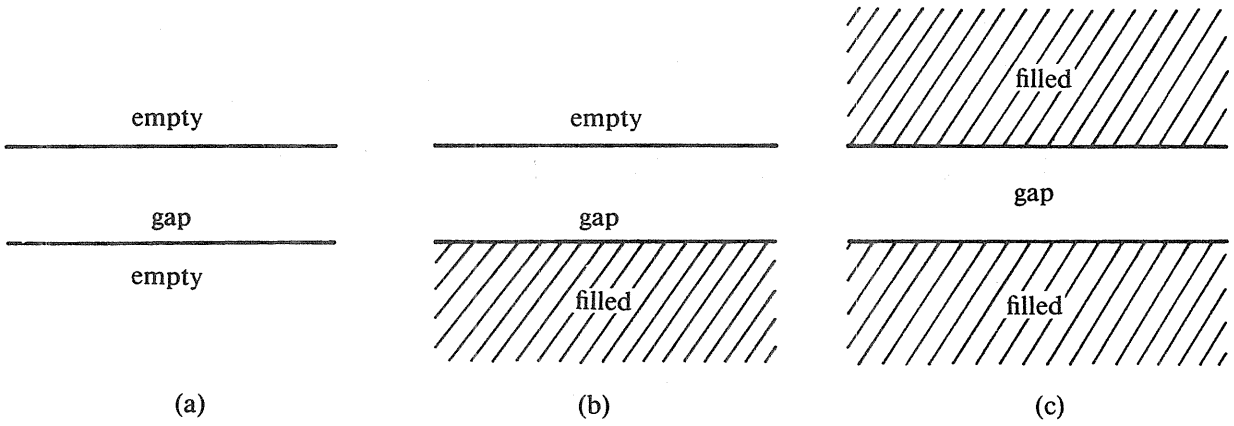


Fig. 2.3 In the purely electron description: (a) the naive electron vacuum, (b) the physical vacuum, and (c) the naive positron vacuum.

is equivalent to removing all positive energy electrons in our description. This will lead to the same physical vacuum as Fig. (2.3b). In the positronic world, the negative infinite constant in H will be interpreted as the energy associated with the filled negative energy positron sea.

Chapter 3

INTERACTIONS AMONG FIELDS

1. An Example of Field Interactions

In reality, particles may be created and annihilated. For instance, in a nucleon-nucleon collision, pions may be created freely. The processes in the box of Fig. 3.1 (a) and of 3.1(b) describe the creation and annihilation of a pion,

$$p \rightarrow \pi^+ + n, \text{ and } p + \pi^- \rightarrow n.$$

We interpret process (a) as a proton destroyed at x, and a neutron and a positive pi meson created at the same place and time. We can describe the above process by an interaction of the form,

$$g\, \bar{\psi}_n\, i\gamma_5\, \psi_p\, (\phi_{\pi^+})^*,$$

where the dimensionless coupling constant g measures the strength of the interaction. Recall that ψ, $\bar{\psi}$ and ϕ are made of creation and annihilation operators of the corresponding particles. We choose the pseudoscalar density $\psi_n i\gamma_5 \psi_p$ because pion is a pseudoscalar particle. This interaction preserves the space inversion transformation.

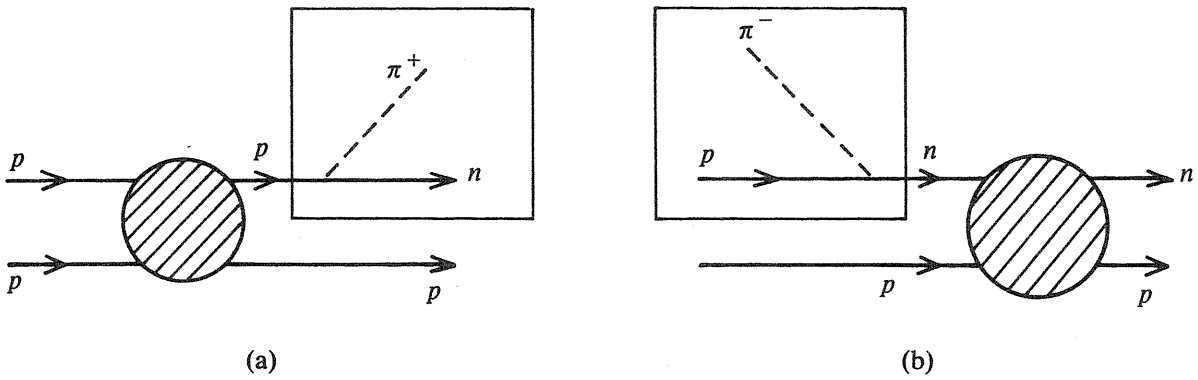


Fig. 3.1 (a) The creation of π^+ and (b) the annihilation of π in nucleon-nucleon scatterings.

Since our purpose is to develop the diagrammatic method, we begin with a simplified version of the above pion-nucleon system. We assume that there is only one kind of nuclean ψ and one kind of pion ϕ, and that their interactions may be described in the Heisenberg picture by

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}} \tag{3.1}$$

with

$$\mathscr{L}_0 = \bar{\psi}\,(i\partial\!\!\!/ - M)\,\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 \tag{3.2}$$

$$\mathscr{L}_{\text{int}} = -g\bar{\psi}i\gamma_5\psi\phi \tag{3.3}$$

where M and μ are the nucleon and pion masses. The minus sign in front of the right-hand side of Eq. (3.3) is a matter of definition. In the simplified model, ϕ is hermitian (real). The bilinear combination $\bar{\psi}\,i\gamma_5\psi$ is also hermitian,

$$\begin{aligned}(\bar{\psi}i\gamma_5\psi)^\dagger &= [\psi_\alpha^\dagger\,(\gamma^0 i\gamma_5)_{\alpha\beta}\psi_\beta]^\dagger \\ &= \psi_\beta^\dagger\,(\gamma^0 i\gamma_5)^*_{\alpha\beta}\,\psi_\alpha \\ &= \psi_\beta^\dagger\,(\gamma^0 i\gamma_5)^{*T}_{\beta\alpha}\,\psi_\alpha \\ &= \psi_\beta^\dagger\,(-i\gamma_5\gamma^0)_{\beta\alpha}\psi_\alpha = \bar{\psi}\,i\gamma_5\psi.\end{aligned} \tag{3.4}$$

Thus, $\mathscr{L}_{\text{int}}$ is hermitian.

We can use the action principle to obtain the field equations. They are

$$(i\partial\!\!\!/ - M - gi\gamma_5\phi)\psi = 0 \tag{3.5}$$

$$(\partial^2 + \mu^2)\,\phi + g\bar{\psi}i\gamma_5\psi = 0. \tag{3.6}$$

We have changed the order of ϕ and ψ in (3.5) to make the equation compact. As we shall see, ϕ and ψ commute at equal time.

The canonical quantization relations are the same as those of the free fields. We have, at equal time,

$$[\phi, \phi] = [\dot{\phi}, \dot{\phi}] = \{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \tag{3.7}$$

$$[\phi(x, t), \dot{\phi}(y, t)] = i\delta^3(x - y), \tag{3.8}$$

$$\{\psi(x, t), \psi^\dagger(y, t)\} = \delta^3(x - y), \tag{3.9}$$

$$[\phi, \psi] = [\phi, \psi^\dagger] = [\dot{\phi}, \psi] = [\dot{\phi}, \psi^\dagger] = 0. \tag{3.10}$$

Equations (3.7)–(3.9) are the same as the quantization rules for free fields. Equation (3.10) is a new condition. It tells us that we should use commutators to describe the quantization rules between a boson and a fermion.

The Hamiltonian density $\mathscr{H}$ is

$$\mathscr{H} = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} \dot{\phi} + \frac{\partial \mathscr{L}}{\partial \dot{\psi}} \dot{\psi} - \mathscr{L}$$

$$= \mathscr{H}_0 + \mathscr{H}_{\text{int}} \tag{3.11}$$

with

$$\mathscr{H}_0 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\gamma \cdot \nabla + M)\psi \tag{3.12}$$

$$\mathscr{H}_{\text{int}} = -\mathscr{L}_{\text{int}} = g\bar{\psi}i\gamma_5\psi\phi. \tag{3.13}$$

The Hamiltonian H is the integral of $\mathscr{H}$. We can show that the Heisenberg equations of motion reproduce the field equations (3.5) and (3.6).

2. Interaction Picture and Dyson's Expansion

In this section, we derive a formal solution to all scattering problems. The expansion of this solution leads to Feynman rules. As in all formal derivations, we have ignored many subtleties and possible complications.

We begin with a Hamiltonian in the Heisenberg representation,

$$H = H_0 + H_{\text{int}}. \tag{3.14}$$

The field operators which we denote collectively as $\{\phi(x, t)\}$ obey the canonical quantization rules and the equations of motion,

$$\dot{\phi}(x, t) = \frac{1}{i}[\phi(x, t), H]. \tag{3.15}$$

We can integrate (3.15) formally to

$$\phi(x, t) = e^{iHt}\,\phi(x, 0)\,e^{-iHt}. \tag{3.16}$$

We can apply (3.16) to any Heisenberg operator $O(t)$ composed of $\{\phi(x, t)\}$,

$$O(t) = e^{iHt}\,O(0)\,e^{-iHt}. \tag{3.17}$$

The state vectors, $|\psi\rangle$, are constant in time in the Heisenberg picture. A typical matrix element is of the form

$$\langle \psi_2 | O(t) | \psi_1 \rangle = \langle \psi_2 | e^{iHt}\,O(0)e^{-iHt} | \psi_1 \rangle. \tag{3.18}$$

Our purpose is to develop a technique to evaluate (3.18).

Dyson introduced an interaction picture such that the field operators in this picture evolute as free fields. The time evolution of an interaction operator O_I is defined as

$$O_I(t) \equiv e^{iH_0t}\,O(0)\,e^{-iH_0t}. \tag{3.19}$$

The operators $O_I(x, t)$ and $O(x, t)$ coincide at the reference time $t = 0$. Obviously, the interaction field operators obey the Heisenberg equations of motion for a free Hamiltonian H_0,

$$\dot{O}_I(t) = \frac{1}{i}[O_I(t), H_0].$$

Since (3.18) is a unitary transformation, the canonical quantization relations are also preserved. Thus, the interaction field variables

$$\phi_I(x, t) \equiv e^{iH_0t}\,\phi(x, 0)\,e^{iH_0t} \tag{3.20}$$

obey the free field equations. We can work out their time evolution explicitly.

In order to preserve the matrix elements, we have to introduce an explicit time dependence into the state vectors

$$|\psi(t)\rangle_I = e^{iH_0 t}\, e^{-iHt}\, |\psi\rangle. \tag{3.21}$$

It is easy to verify that the matrix element in the interaction picture agrees with that of the Heisenberg picture:

$$\begin{aligned} {}_I\langle \psi_2(t)|O_I(t)|\psi_1(t)\rangle_I &= \langle \psi_2|e^{iHt}\, e^{-iH_0 t}\left(e^{iH_0 t}\, O(0)\, e^{-iH_0 t}\right) \\ &\quad \times e^{iH_0 t}\, e^{-iHt}\, |\psi_1\rangle \\ &= \langle \psi_2|e^{iHt}\, O(0)\, e^{-iHt}|\psi_1\rangle \\ &= \langle \psi_2|O(t)|\psi_1\rangle. \end{aligned} \tag{3.22}$$

The time dependence of $|\psi(t)\rangle_I$ obeys

$$\begin{aligned} \frac{\partial}{\partial t}|\psi(t)\rangle_I &= e^{iH_0 t}\,(iH_0 - iH)\, e^{-iHt}\, |\psi\rangle \\ &= e^{iH_0 t}\,(-iH_{\text{int}})\, e^{-iH_0 t}\, |\psi(t)\rangle_I \\ &= -iH_I(t)\,|\psi(t)\rangle_I \end{aligned} \tag{3.23}$$

where

$$H_I(t) \equiv e^{iH_0 t}\, H_{\text{int}}\, e^{-iH_0 t} \tag{3.24}$$

is the interaction Hamiltonian in the interaction picture.

It is convenient to describe the time dependence of the interaction representation states through a U operator, introduced below. The U operator, also known as the evolution operator, is defined by

$$|\psi(t_1)\rangle_I = U(t_1, t_2)\,|\psi(t_2)\rangle_I. \tag{3.25}$$

From (3.23) and (3.25), we find that the U-operator obeys

$$\frac{\partial}{\partial t_1}\, U(t_1, t_2) = -iH_I(t_1)\, U(t_1, t_2) \tag{3.26}$$

and

$$U(t_1, t_1) = I, \text{ identity.} \tag{3.27}$$

Obviously, we also have

$$U(t_1, t_2)\, U(t_2, t_3) = U(t_1, t_3) \tag{3.28}$$

$$U(t_1, t_2)^{-1} = U^{\dagger}(t_1, t_2) = U(t_2, t_1). \tag{3.29}$$

To solve (3.26), we first turn it into an integral equation

$$U(t_1, t_2) = I - i \int_{t_2}^{t_1} dt\, H_I(t)\, U(t, t_2). \tag{3.30}$$

It is easy to see that both (3.26) and (3.27) follow from (3.30). Next, we iterate (3.30) twice to give

$$\begin{aligned} U(t_1, t_2) &= I + (-i) \int_{t_1}^{t_2} dt'\, H_t(t') \\ &\quad + (-i)^2 \int_{t_2}^{t_1} dt' \int_{t_2}^{t'} dt''\, H_I(t')\, U(t', t_2) \\ &= I + (-i) \int_{t_2}^{t_1} dt'\, H_I(t') \\ &\quad + (-i)^2 \int_{t_2}^{t_1} dt' \int_{t_2}^{t'} dt''\, H_I(t')\, H_I(t'') \\ &\quad + (-i)^3 \int_{t_2}^{t_1} dt' \int_{t_2}^{t'} dt'' \int_{t_2}^{t''} dt'''\, H_I(t')\, H_I(t'')\, H_I(t''')\, U(t''', t_2). \end{aligned} \tag{3.31}$$

Note that the products of H_I in the integrands always follow the time sequence. We define a time-ordered product of two operators as

$$\begin{aligned} T(A(t_1)\, B(t_2)) &= A(t_1)\, B(t_2),\ t_1 > t_2 \\ &= B(t_2)\, A(t_1),\ t_2 > t_1. \end{aligned} \tag{3.32}$$

We can define the time-ordered product of n operators in a similar way. We can extend the ranges of integration in (3.31) to the full ranges t_2 to t_1 provided that we introduce the time-ordered products of H_I as the integrand and include a $1/n!$ compensation factor, giving

$$U(t_1, t_2) = \sum_n \frac{(-i)^n}{n!} \int_{t_1}^{t_2} dt'\, dt'' \ldots dt^{(n)}\, T(H_I(t') \ldots H_I(t^{(n)}))$$

$$= T \exp\left(-i \int_{t_1}^{t_2} dt\, H_I(t)\right). \tag{3.33}$$

We define the evolution operator which relates the states of distant past ($t = -\infty$) to the states of distant future ($t = \infty$) as the scattering matrix (S-matrix). The S-matrix has the form

$$S \equiv U(\infty, -\infty) = T \exp\left(-i \int_{-\infty}^{\infty} dt\, H_I(t)\right)$$

$$= T \exp\left(-i \int_{-\infty}^{\infty} d^4x\, \mathscr{H}_I(x, t)\right). \tag{3.34}$$

3. S-Matrix, In-states, and Out-states

We may describe the general scattering phenomena by a set of particles coming in at $t \to -\infty$ and another set of particles going out at $t \to +\infty$, as shown in Fig. 3.2. We usually choose the mass parameters in H_0 to coincide with the observed masses of the particles.

For the incoming particles, we can use the free particle states to form a complete set of states. From these plane wave solutions, we can construct a set of free fields, $\phi_{\text{in}}(x, t)$, which we shall refer to as in-fields. The in-fields obey the free-particle equations and the canonical quantization relations. Indeed, if we choose our reference time at $t = -\infty$, we may identify these in-fields as our interaction picture field variables. For widely separated particles, we only need to know whether we can

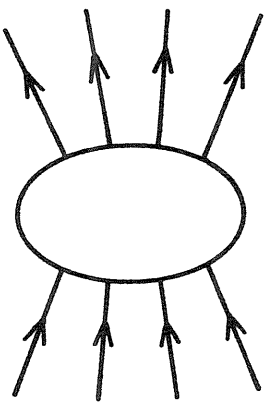


Fig. 3.2 A general scattering process.

approximate a one-particle state by an incoming free particle state. A multiparticle state is the product of one particle states.

There is a question about whether we can approximate our incoming particles by a set of free particles. The answer to this question is quite subtle. Note that even an isolated physical particle is still subject to the self-interaction. When we try to create a particle from the vacuum by $\phi(x, t)$, we may also create some other excited states. Since we normalize both ϕ and ϕ_{in} by the same canonical quantization rules, the ability of ϕ to create or annihilate a single particle is less than that of the free field ϕ_{in}. Thus, in order to give the same matrix element, we need to introduce a $\sqrt{Z}$ (≤ 1) factor when we replace the one-particle state by the free one-particle state. At the moment, we shall temporarily forget about these $\sqrt{Z}$ factors in the incoming and outgoing particles, and replace the multiparticle states at $t \to +\infty$ by free particle states. Then we can express the S-matrix elements as

$$S_{fi} = {}_{\text{in}}\left\langle k'_1, k'_2, \ldots \left| T\exp\left(-i\int_{-\infty}^{\infty} dt\, H_I(t)\right)\right| k_1, k_2, \ldots \right\rangle_{\text{in}}. \quad (3.35)$$

4. Wick's Theorem

In evaluating the S-matrix, we need some rules to simplify the time-ordered integrals. Wick established an expansion method which now bears his name. For definiteness, we consider the Hamiltonian,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}, \quad (3.36)$$

$$\mathcal{H}_0 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\mu^2\phi^2$$

$$+ \bar{\psi}\,(i\boldsymbol{\gamma}\cdot\nabla + M)\,\psi, \quad (3.37)$$

$$\mathcal{H}_I = g\,\bar{\psi}\,i\gamma_5\,\psi\phi. \quad (3.38)$$

In (3.36)–(3.38), field variables ϕ, ψ and $\bar{\psi}$ are in the interaction representation, and obey the free field equations. We have ignored the subscript I for notational simplicity. We can expand these field variables as linear combinations of the creation and the annihilation operators,

$$\phi(x, t) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}}\left(e^{-ik\cdot x}a_k + e^{ik\cdot x}a_k^\dagger\right) \quad (3.39)$$

$$\psi(x,t) = \int d^3p \sqrt{\frac{M}{(2\pi)^3 E_p}} \left[b(p,s)\, u(p,s)\, e^{-ip\cdot x} + d^\dagger(p,s)\, v(p,s)\, e^{ip\cdot x} \right]. \tag{3.40}$$

Wick introduced two kinds of operator orderings.

(1) Normal ordering. This ordering is denoted by a pair of colons. (: . . . :) Let field A_i be a linear combination of the creation and the annihilation operators. We define the normal-ordered product, $:A_1 A_2 \ldots A_n:$, as the product of field operators A_i arranged with all the creation operators on the right of all the annihilation operators. In the presence of fermionic operators, we also need to include a factor $(-1)^N$ where N denotes the number of permutations of the fermionic operators in the rearrangement.

Examples

$$: a_{k_1} a^\dagger_{k_2} a_{k_3} a^\dagger_{k_4} : \equiv a^\dagger_{k_2} a^\dagger_{k_4} a_{k_1} a_{k_3},$$

$$: a_{k_1} a^\dagger_{k_2} b_{p_1} b^\dagger_{p_2} : \equiv -a^\dagger_{k_2} b^\dagger_{p_2} a_{k_1} b_{p_1}.$$

The advantage of the normal orderings is that once we reduce a product of operators into the normal orderings, we can read off its matrix element trivially. For instance, the only nonvanishing matrix element for

$$: a^\dagger_{k'_1} a^\dagger_{k'_2} \ldots a^\dagger_{k'_m} a_{k_1} \ldots a_{k_n} :$$

is between states $\langle k'_1, \ldots, k'_m |$ and $| k_l, \ldots, k_n \rangle$.

(2) Time ordering. We extend the time-ordering definition to individual field operators ϕ, ψ, $\bar{\psi}$. In arranging the order of the fermionic operators, we also include a $(-1)^N$ factor for the permutations of fermionic operators, just as in the normal-ordered product.

We shall work out a few simple cases of reducing the time-ordered products into the normal-ordered products. The time-ordered product of a single field variable is trivial, and is the same as the normal-product,

$$T(\phi(x)) = :\phi(x):,\ T(\psi(x)) = :\psi(x):,\ T(\bar{\psi}(x)) = :\bar{\psi}(x):.$$

The time-ordered products of two field variables begin to give nontrivial

results. For the product of two ϕ's, we have

$$T(\phi(x)\,\phi(y)) = :\phi(x)\,\phi(y): + \overbrace{\phi(x)\,\phi(y)} \tag{3.41}$$

where $\overbrace{\phi(x)\,\phi(y)}$ is a number called the contraction of $\phi(x)$ and $\phi(y)$ also known as a propagator. For $x^0 > y^0$, we have

$$\begin{aligned} T(\phi(x)\,\phi(y)) &= \phi(x)\,\phi(y) \\ &= \int \frac{d^3k}{\sqrt{2\omega_k\,(2\pi)^3}} \int \frac{d^3l}{\sqrt{2\omega_l\,(2\pi)^3}} (e^{-ik\cdot x} a_k + e^{ik\cdot x} a_k^\dagger) \\ &\quad \times (e^{-il\cdot y} a_l + e^{il\cdot y} a_l^\dagger) \\ &= \int \frac{d^3k}{\sqrt{2\omega_k\,(2\pi)^3}} \int \frac{d^3l}{\sqrt{2\omega_l\,(2\pi)^3}} [e^{-ik\cdot x - il\cdot y}\, a_k a_l \\ &\quad + e^{ik\cdot x - il\cdot y}\, a_k^\dagger\, a_l + e^{-ik\cdot x + il\cdot y}\, a_k\, a_l^\dagger + e^{ik\cdot x + il\cdot y}\, a_k^\dagger\, a_l^\dagger] \\ &= :(\phi(x)\,\phi(y):\, + \int \frac{d^3k\,d^3l}{\sqrt{2\omega_k 2\omega_l}(2\pi)^3} e^{-ik\cdot x + il\cdot y}\, \delta^3(k-l). \end{aligned}$$

Hence, for $x^0 > y^0$, we obtain

$$\overbrace{\phi(x)\,\phi(y)} = \int \frac{d^3k}{2\omega_k(2\pi)^3} e^{-ik\cdot(x-y)} \,. \tag{3.42}$$

Similarly, for $y^0 > x^0$, we have

$$\overbrace{\phi(x)\,\phi(y)} = \int \frac{d^3k}{2\omega_k(2\pi)^3} e^{-ik\cdot(y-x)} \,. \tag{3.43}$$

We can combine these two equations to give

$$\begin{aligned} \overbrace{\phi(x)\,\phi(y)} &\equiv i\Delta_F(x-y) \\ &= \int \frac{d^3k}{2\omega_k(2\pi)^3} [e^{-ik\cdot(x-y)}\,\theta(x^0-y^0) + e^{-ik\cdot(y-x)}\,\theta(y^0-x^0)]. \end{aligned} \tag{3.44}$$

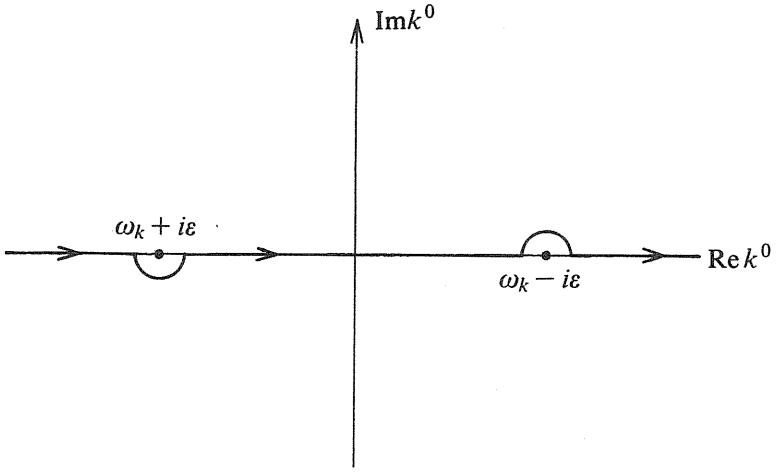


Fig. 3.3 The complex k-plane.

It is possible to show that Eq. (3.44) leads to

$$i\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2-\mu^2+i\varepsilon} \tag{3.45}$$

where ε is an infinitesimal positive number. We wish to add some clarification of the notation. In (3.44), k^0 in the exponent is the on-shell value $k^0 = \omega_k \equiv \sqrt{k^2+\mu^2}$. In (3.45), however, k^0 in the exponent is an integration variable. To see that (3.44) and (3.45) are equivalent, we integrate out the k^0-integration in (3.45). The denominator of (3.45) has 2 poles located at $k^0 = \pm(\omega_k - i\varepsilon)$. (See Fig. 3.3). For $x^0 > y^0$, we can only enclose the contour from the lower half plane, and pick up the pole contribution at $k^0 = \omega_k$, obtaining

$$\begin{aligned} i\Delta_F(x-y) &= \int \frac{d^3k}{(2\pi)^4} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \int dk^0 e^{-ik^0(x^0-y^0)} \\ &\quad \times \frac{i}{(k^0-\omega_k+i\varepsilon)(k^0+\omega_k-i\varepsilon)} \\ &= \int \frac{d^3k}{(2\pi)^4} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} e^{-i\omega_k\cdot(x^0-y^0)} \frac{(-2\pi i)i}{\omega_k+\omega_k} \\ &= \int \frac{d^3k}{2\omega_k(2\pi)^3} e^{-ik\cdot(x-y)}, \; k^0 = \omega_k. \end{aligned} \tag{3.46}$$

Thus, (3.45) agrees with (3.44) for $x^0 > y^0$. Similarly we can show that (3.45) also reproduces (3.44) for $y^0 > x^0$.

The time-ordered products for a ϕ and a ψ (or $\bar{\psi}$) for two ψ's and for two $\bar{\psi}$'s all give trivial results,

$$T(\phi(x)\,\phi(y)) = :\phi(x)\,\psi(y): \tag{3.47}$$

$$T(\psi(x)\,\psi(y)) = :\psi(x)\,\psi(y): \tag{3.48}$$

$$T(\bar{\psi}(x)\,\bar{\psi}(y)) = :\bar{\psi}(x)\,\bar{\psi}(y):. \tag{3.49}$$

The time-ordered product for a ψ and a $\bar{\psi}$ gives

$$T(\psi_\alpha(x)\,\bar{\psi}_\beta(y)) = :\psi_\alpha(x)\,\bar{\psi}_\beta(y): + \overbrace{\psi_\alpha(x)\,\bar{\psi}_\beta(y)}. \tag{3.50}$$

Note that the fermion contraction (fermion propagator) is always from $\bar{\psi}$ to a ψ. We can work out the fermion contraction term straightforwardly as

$$\begin{aligned}\overbrace{\psi_\alpha(x)\,\bar{\psi}_\beta(y)} &\equiv iS_F(x-y)_{\alpha\beta} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i(\not{p}+M)}{p^2 - M^2 + i\varepsilon} \\ &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{\not{p} - M + i\varepsilon}\ .\end{aligned} \tag{3.51}$$

Now, we encounter the general situation. We shall state the result without proving it.

Wick Theorem We can express any time-ordered product of free fields, $T(A_1A_2\ldots A_n)$, as summations over the normal-ordered products generated by all possible contractions.

We shall use the following examples to illustrate the theorem. For the product of four pion operators, we have

$$\begin{aligned}T(\phi_1\phi_2\phi_3\phi_4) = {} & :\phi_1\phi_2\phi_3\phi_4: + :\overbrace{\phi_1\phi_2}\phi_3\phi_4: \\ & + :\overbrace{\phi_1\phi_2\phi_3}\phi_4: + \ldots + :\phi_1\phi_2\overbrace{\phi_3\phi_4}: \\ & + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4} + \overbrace{\phi_1\overbrace{\phi_2\phi_3\phi_4}}\end{aligned}$$

$$= :\phi_1\phi_2\phi_3\phi_4: + i\Delta_F(1,2):\phi_3\phi_4: + i\Delta_F(1,3):\phi_2\phi_4:$$

$$+ \ldots + i\Delta_F(3,4):\phi_1\phi_2: + i\Delta_F(1,2)\,i\Delta_F(3,4)$$

$$+ i\Delta_F(1,3)\,i\Delta_F(2,4) + i\Delta_F(1,4)\,i\Delta_F(2,3). \tag{3.52}$$

For the product of four fermion operators (2ψ and $2\bar{\psi}$), we have

$$T(\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4) = :\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4: + :\overset{\frown}{\bar{\psi}_1}\psi_2\bar{\psi}_3\psi_4:$$

$$+ :\overset{\frown}{\bar{\psi}_1\psi_2}\bar{\psi}_3\psi_4: + :\bar{\psi}_1\overset{\frown}{\psi_2}\bar{\psi}_3\psi_4:$$

$$+ :\bar{\psi}_1\psi_2\overset{\frown}{\bar{\psi}_3}\psi_4: + \overset{\frown}{\bar{\psi}_1}\psi_2\overset{\frown}{\bar{\psi}_3}\psi_4 + \overset{\frown}{\bar{\psi}_1\psi_2\bar{\psi}_3}\psi_4$$

$$= :\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4: + (-iS_F)_{21}:\bar{\psi}_3\psi_4:$$

$$+ (-iS_F)_{41}:\psi_2\bar{\psi}_3: + (iS_F)_{23}:\bar{\psi}_1\psi_4: + (-iS_F)_{43}:\bar{\psi}_1\psi_2:$$

$$+ (-iS_F)_{21}(-iS_F)_{43} + (iS_F)_{23}(-iS_F)_{41}. \tag{3.53}$$

We can work out other products in a similar way.

5. Second order processes

We shall work out the reduction of second order processes explicitly. The second order expansion of the S-matrix,

$$S = T\exp\left(-i\int_{-\infty}^{\infty} dt\, H_I(t)\right) = T\exp\left(-i\int_{-\infty}^{\infty} d^4x\, \mathscr{H}_I(x)\right)$$

is

$$S^{(2)} = \frac{(-i)^2}{2!}\int_{-\infty}^{\infty} d^4x_1\, d^4x_2\, T(\mathscr{H}_I(x_1)\,\mathscr{H}_I(x_2)). \tag{3.54}$$

We choose $\mathscr{H}_I$ to be $g\bar{\psi}i\gamma_5\psi\phi$, and obtain

$$S^{(2)} = \frac{(-ig)^2}{2!}\int_{-\infty}^{\infty} d^4x_1\, d^4x_2\, T(\bar{\psi}(x_1)\,i\gamma_5\,\psi(x_1)\,\phi(x_1)$$

$$\times\, \bar{\psi}(x_2)\,i\gamma_5\,\psi(x_2)\,\phi(x_2)). \tag{3.55}$$

Now we concentrate on the reduction of the time-ordered product. We also introduce some graphic rules (See Fig. 3.4 below). We represent the interaction $\mathscr{H}_I$ by a vertex with $\bar{\psi}$ as an outgoing line, ψ as an incoming line, and ϕ as a dashed line.

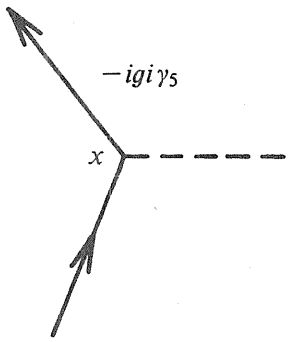


Fig. 3.4a A vertex.

We represent a pion contraction $\overbrace{\phi(x_1)\phi(x_2)}$ by

Fig. 3.4b Pion contraction.

and a nucleon contraction $\overbrace{\psi(x_1)\,\bar{\psi}(x_2)}$ by

Fig. 3.4c Nucleon contraction.

Note that $\overbrace{\psi(x_1)\,\bar{\psi}(x_2)}$ is represented by a line with a direction pointing from x_2 to x_1. This direction line describes the flow of a nucleonic charge from x_2 to x_1. Since the nucleonic number is conserved, these nucleon lines form continuous trajectories from $t = -\infty$ to $t = +\infty$.

With the help of the above graphic rules, Wick's theorem gives the following expansions:

(1) $: \bar{\psi}_1 i\gamma_5 \psi_1 \phi_1 \bar{\psi}_2 i\gamma_5 \psi_2 \phi_2 :$

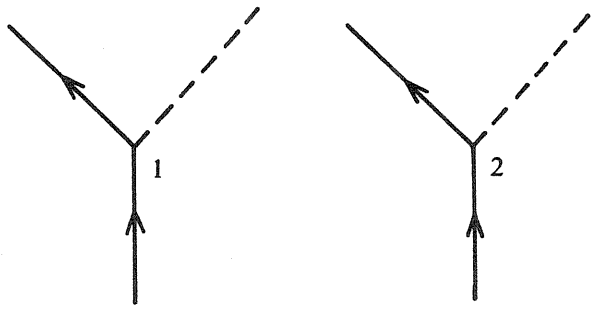


Fig. 3.5 Two noninteracting vertices.

This term represents two non-interacting vertices. This process is forbidden by the energy-momentum conservation and the mass-shell relation.

(2) : $\bar{\psi}_1\, i\gamma_5\, \psi_1\, \overbrace{\phi_1\, \bar{\psi}_2}\, i\gamma_5\, \psi_2\, \phi_2$:

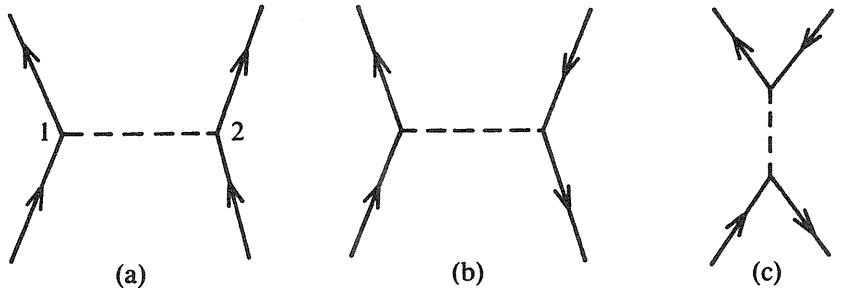


Fig. 3.6 (a)–(b) One-pion exchange processes, and (c) N-$\bar{N}$ annihilation processes.

Figures 3.6a and 3.6b describe one-pion exchange processes, and 3.6c describes a nucleon-antinucleon annihilation process. Note that the field variables not contracted are represented by external particles.

(3) : $\bar{\psi}_1\, i\gamma_5\, \overbrace{\psi_1\, \phi_1}\, \bar{\psi}_2\, i\gamma_5\, \psi_2\, \phi_2$:

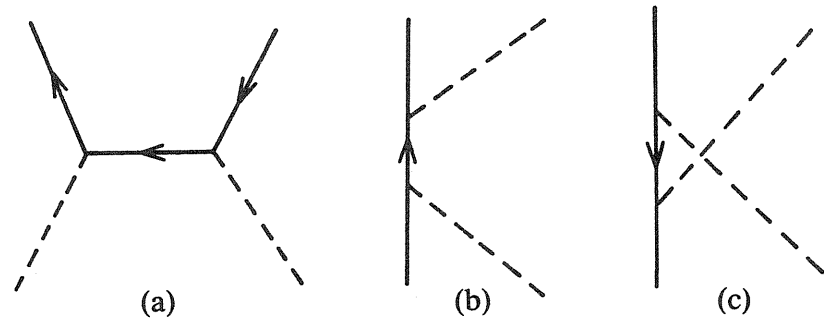


Fig. 3.7 Several possible processes with 2 external pions and 2 external nucleons or antinucleons.

Figures 3.7 (a)–(c) describe some possible processes including $\pi\pi \to N\bar{N}$, πN scattering and $\pi\bar{N}$ scattering.

(4) $: \bar{\psi}_1 \, i\gamma_5 \, \psi_1 \, \phi_1 \, \bar{\psi}_2 \, i\gamma_5 \, \psi_2 \, \phi_2 :$

Fig. 3.8 Nucleon self-energy.

Figure 3.8 represents a self-energy contribution to the nucleon.

(5) $: \bar{\psi}_1 \, i\gamma_5 \, \psi_1 \, \phi_1 \, \bar{\psi}_2 \, i\gamma_5 \, \psi_2 \, \phi_2 :$

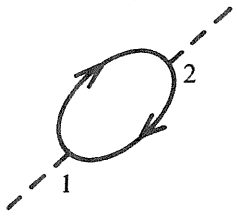


Fig. 3.9 Pion self-energy.

Figure 3.9 represents a self-energy contribution to the pion.

(6) $: \bar{\psi}_1 \, i\gamma_5 \, \psi_1 \, \phi_1 \, \bar{\psi}_2 \, i\gamma_5 \, \psi_2 \, \phi_2 :$

Fig. 3.10 Vacuum diagram.

Figure 3.10 is a vacuum diagram.

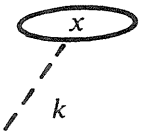


Fig. 3.11 An external leg.

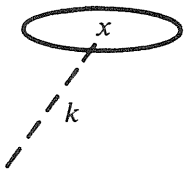
In order to obtain the S-matrix, we also need to know how to evaluate the external lines. Since the field operators have already been normal-ordered, they annihilate and create particles directly without further interaction. Consider the external leg described in Fig. 3.11. Since

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} [e^{-ik\cdot x} a_k + e^{ik\cdot x} a_k^\dagger],$$

and since only the first term describing the annihilation process contributes, we obtain the amplitude associated with an initial pion in Fig. 3.11 as

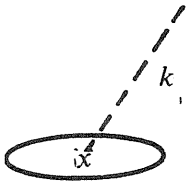
$$\langle 0|\phi(x)|k\rangle = \frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3 2\omega_k}}.$$

We summarize the amplitudes associated with various external legs in Fig. 3.12.



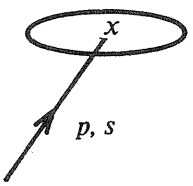
$$\frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ik\cdot x}$$

(a)



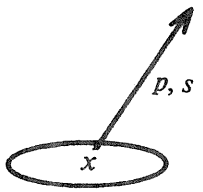
$$\frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{ik\cdot x}$$

(b)



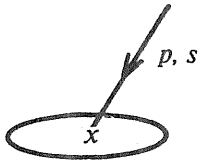
$$\sqrt{\frac{M}{(2\pi)^3 E_p}}\, u(p, s)\, e^{-ip\cdot x}$$

(c)



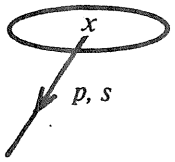
$$\sqrt{\frac{M}{(2\pi)^3 E_p}}\,\bar{u}(p,s)\,e^{ip\cdot x}$$

(d)



$$\sqrt{\frac{M}{(2\pi)^3 E_p}}\,v(p,s)\,e^{ip\cdot x}$$

(e)



$$\sqrt{\frac{M}{(2\pi)^3 E_p}}\,\bar{v}(p,s)\,e^{-ip\cdot x}$$

(f)

Fig. 3.12 (a)–(f) Amplitudes associated with various external legs.

The rules that we have developed here form the basis of the position space Feynman rules.

We now work out a few second order amplitudes in detail.

6. Explicit Amplitudes

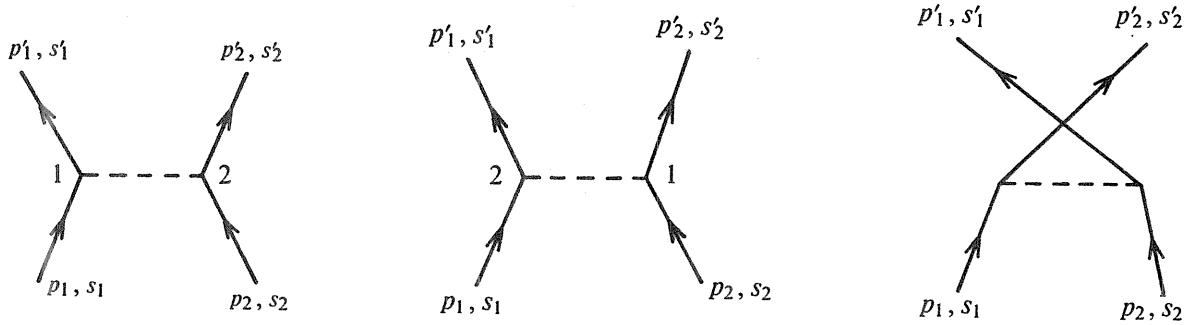


Fig. 3.13 Nucleon-nucleon scattering with one-pion exchanged.

Consider Figs. 3.13. They represent the nucleon-nucleon scatterings with a pion being exchanged. We consider the first two diagrams as described in Figs. 3.13a and 3.13b. The amplitude for Fig. 3.13a is

$$A_{(a)} = \frac{(-ig)^2}{2!} \int d^4x_1\, d^4x_2 \sqrt{\frac{M}{(2\pi)^3 E_1'}}\, e^{ip_1' \cdot x_1}$$
$$\times \sqrt{\frac{M}{(2\pi)^3 E_1}}\, e^{-ip_1 \cdot x_1} \sqrt{\frac{M}{(2\pi)^3 E_2'}}\, e^{ip_2' \cdot x_2}$$
$$\times \sqrt{\frac{M}{(2\pi)^3 E_2}}\, e^{-ip_2 \cdot x_2}\, \bar{u}(p_1', s_1')\, i\gamma_5\, u(p_1, s_1)$$
$$\times \bar{u}(p_2', s_2')\, i\gamma_5\, u(p_2, s_2)\, i\Delta_F(x_1 - x_2). \qquad (3.56)$$

We can obtain the amplitude for Figs. 3.13b by interchanging x_1 and x_2 in (3.56). Since x_1 and x_2 are integration variables, the interchange of x_1 and x_2 does not affect the amplitude. Hence, the total amplitude of Fig. 3.13a and 3.13b is twice that of Eq. (3.56). This extra factor of 2 cancels the 1/2! in (3.56). In the future, we shall only draw a single diagram to describe both Figs. 3.13a and 3.13b, and ignore the 1/2! factor in (3.56). Note that Fig. 3.13c represents a different diagram. Its amplitude should be computed and included separately.

We need to explain how to deal with the Dirac indices. In the present case, we have constructed two scalars out of matrix multiplications. The first scalar begins with $\bar{u}(p_1', s_1')$ of the final particle p_1', is followed by $i\gamma_5$ of the vertex 1, and ends with $\bar{u}(p_1, s_1)$ of the initial particle p_1. In short, we follow the nucleon line along particle p'_1, vertex 1, and particle p_1. For the second scalar, we follow the nucleon line along particle p_2', vertex 2, and particle p_2. The general rule of Dirac matrix multiplications is to follow the nucleon lines (arrows) backward.

Fig. 3.14 Pion self-energy diagram.

Next, we consider Fig. 3.14, a pion self-energy diagram. Usually, this diagram appears in the middle of another diagram. We now concentrate on the nucleon loop. The double-nucleon contraction gives

$$(\bar{\psi}_1)_\alpha (i\gamma_5)_{\alpha\beta} (\psi_1)_\beta (\bar{\psi}_2)_\gamma (i\gamma_5)_{\gamma\delta} (\psi_2)_\delta$$

$$= (-1)\, (i\gamma_5)_{\alpha\beta} (i\gamma_5)_{\gamma\delta}\, iS_F (x_1 - x_2)_{\beta\gamma}\, iS_F (x_2 - x_1)_{\delta\alpha}$$

$$= (-1)\, \mathrm{Tr}\, [i\gamma_5\, iS_F (x_1 - x_2)\, i\gamma_5\, iS_F (x_2 - x_1)]. \tag{3.57}$$

Both the (-1) factor and the Trace over Dirac indices in (3.57) are significant, and are general properties of the fermion loop.

The amplitude for the self-energy diagram is

$$A_{(3.14)} = (-ig)^2 \int d^4x_1\, d^4x_2\, (-1)\, \mathrm{Tr}\, [i\gamma_5\, iS_F (x_1 - x_2)\, i\gamma_5\, iS_F (x_2 - x_1)]. \tag{3.58}$$

Note that the 1/2! factor is cancelled by the contribution via $x_1 \leftrightarrow x_2$.

From this diagram, we learned two things. (1) For the Dirac indices concerned, an internal fermion loop represents a trace. Since a trace is invariant under the cyclic permutation of their elements, we can begin the Dirac index at any point on the loop. (2) There is an extra (-1) factor associated with a fermion loop.

The self-energy contribution changes the physical mass of the pion, and introduce a renormalization constant $\sqrt{Z}$ to pions. We shall study the pion and the nucleon self-energy diagrams in detail in the future.

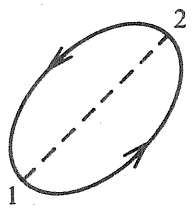


Fig. 3.15 A vacuum diagram.

We shall conclude this section by examining the vacuum diagram, Fig. 3.15. The amplitude of this vacuum diagram is

$$A_{(3.15)} = \frac{(-ig)^2}{2!} \int d^4x_1\, d^4x_4\, (-1)\, \mathrm{Tr}\, [iS_F (x_2 - x_1)\, i\gamma_5 \times iS_F (x_1 - x_2)\, i\gamma_5]\, i\Delta_F (x_2 - x_1). \tag{3.59}$$

There are several important aspects of the vacuum diagrams that we wish to discuss.

(1) The factor 1/2! in (3.59) does not get cancelled because the interchange of $x_1 \leftrightarrow x_2$ does not lead to a distinct diagram. We refer this leftover 1/2 factor as a symmetry factor. The presence of symmetry factors is crucial for the exponentiation to be discussed below.

(2) The sum of connected and disconnected vacuum amplitudes exponentiates as indicated in Fig. 3.16. Only the distinct connected vacuum diagrams contribute to the exponent.

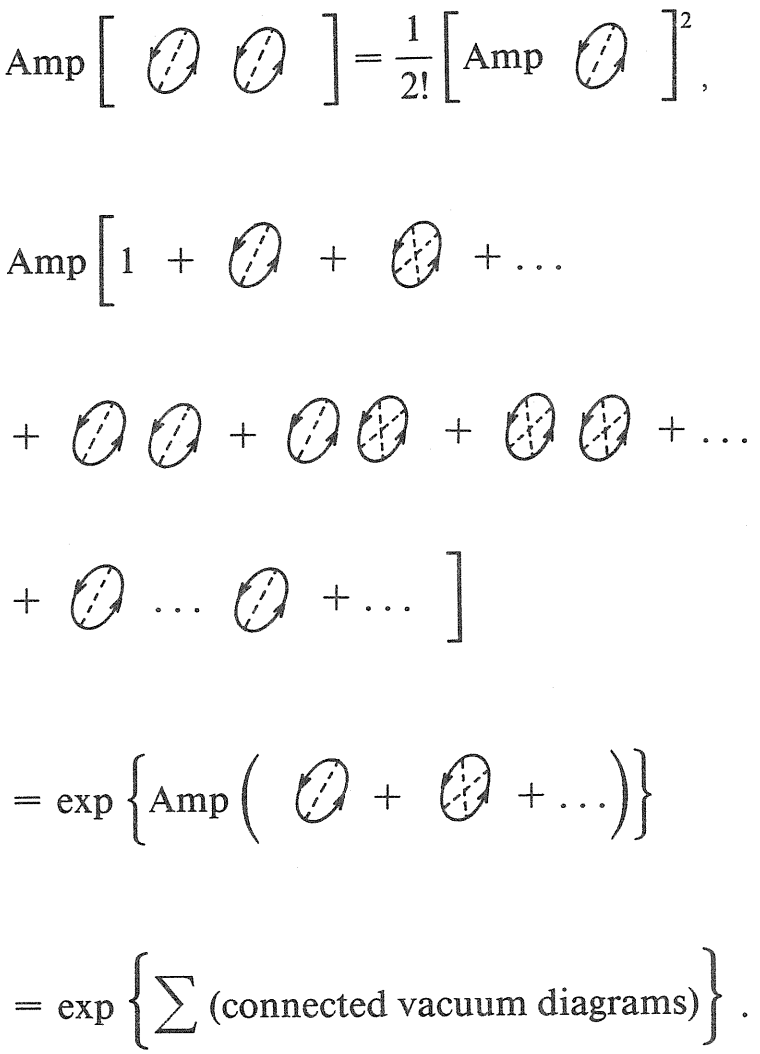


Fig. 3.16 (a) The amplitude of 2 disconnected vacuum diagrams is related to the product of the individual amplitudes, and (b) The total vacuum amplitude exponentiates.

(3) The integrand of (3.57) depends only on $x_1 - x_2$. Hence, the connected vacuum amplitude in Fig. 3.15 is proportional to $\int d^4x = VT$. In addition, if the vacuum is stable, the connected vacuum diagrams are always purely imaginary. We can write the sum of connected vacuum diagrams as $-iVT\Delta\varepsilon$ and the sum of all

vacuum diagrams, connected or disconnected, as $\exp(-iVT\Delta\varepsilon)$. The factor $\Delta\varepsilon$ measures the changes of vacuum energy density.

(4) The vacuum diagrams are also present in any scattering process, and lead to an extra phase factor $\exp(-iVT\Delta\varepsilon)$. If we measure the energy of our states from and above the new ground state energy $V\Delta\varepsilon$, we can forget about all these vacuum diagrams altogether. In the future, we shall ignore the vacuum diagrams, except when we need to compute the change of the ground state energy explicitly.

Chapter 4

FEYNMAN DIAGRAMS AND CROSS SECTIONS

1. Position-space Feynman Rules

We can summarize the results in the last chapter as

(a) Draw all the diagrams made of external lines, propagators (internal lines) and vertices.

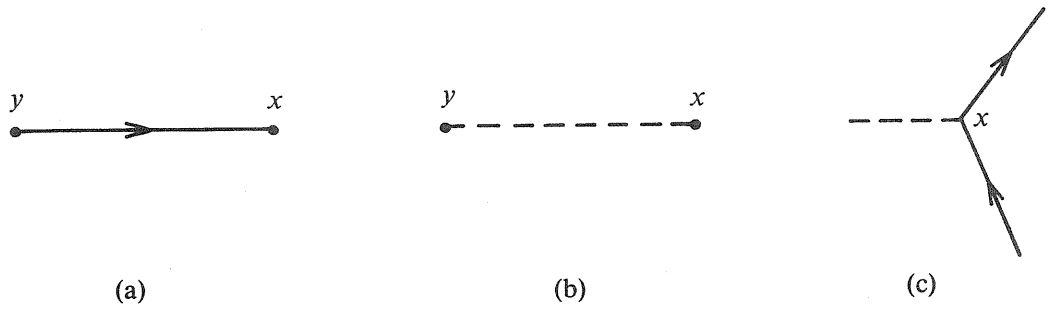


Fig. 4.1 Propagators and vertex in πN interactions.

(b) Represent the nucleon line in Fig. 4.1a by $iS_F(x - y)$, and the meson line in Fig. 4.1b by $i\Delta_F(x - y)$.

(c) Represent the vertex in Fig. 4.1c by $(-ig)i\gamma_5$.

(d) Represent the external particles by factors given in Fig. 3.12 of the previous chapter.

(e) Follow the Dirac matrices along a nucleon line, starting from either an outgoing nucleon, or an incoming antinucleon. For a closed nucleon loop, take a trace over the Dirac matrices and insert a factor (-1).

(f) If the permutation of all n vertices leads to $n!/S$ distinct diagrams, then include a symmetry factor $1/S$. Parameter S is the order of the symmetry subgroup of the permutation group which leaves the diagram invariant.

(g) integrate over all internal coordinates, and sum over the perturbation order n.

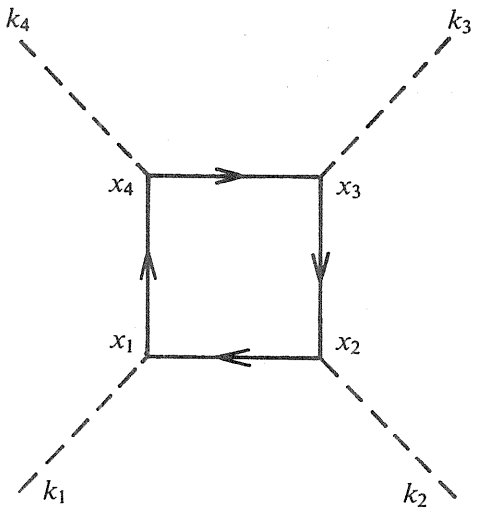


Fig. 4.2 A pion-pion scattering amplitude with a box diagram.

As an example, we can write down the position-space Feynman amplitude for Fig. 4.2 (the box diagram) as

$$A = (-ig)^4 \int d^4x_1\, d^4x_2\, d^4x_3\, d^4x_4 \sqrt{\frac{1}{(2\pi)^3 2\omega_1}}\, e^{-ik_1\cdot x_1}$$

$$\times \sqrt{\frac{1}{(2\pi)^3 2\omega_2}}\, e^{-ik_2\cdot x_2} \sqrt{\frac{1}{(2\pi)^3 2\omega_3}}\, e^{ik_3\cdot x_3} \sqrt{\frac{1}{(2\pi)^3 2\omega_4}}\, e^{ik_4\cdot x_4}$$

$$\times (-1)\,\mathrm{Tr}\,[iS_F(x_1-x_2)\, i\gamma_5\, iS_F(x_2-x_3)\, i\gamma_5$$

$$\times iS_F(x_3-x_4)\, i\gamma_5\, iS_F(x_4-x_1)\, i\gamma_5]. \tag{4.1}$$

Note that the symmetry factor S equals 1.

2. Momentum-space Feynman rules

We may fourier transform the Feynman amplitudes to achieve simpler results. Consider the amplitude given in (4.1). We substitute the fourier representations of $iS_F(x_i - x_j)$ into (4.1) to give

$$A = (-ig)^4 \sqrt{\frac{1}{(2\pi)^4 2\omega_1}} \sqrt{\frac{1}{(2\pi)^4 2\omega_2}} \sqrt{\frac{1}{(2\pi)^4 2\omega_3}} \sqrt{\frac{1}{(2\pi)^4 2\omega_4}}$$

$$\times \int d^4x_1 d^4x_2 d^4x_3\, d^4x_4\, e^{-ik_1\cdot x_1} e^{-ik_2\cdot x_2} e^{ik_3\cdot x_3} e^{ik_4\cdot x_4}$$

$$\times \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4} e^{-ip_1(x_1-x_2)} e^{-ip_2(x_2-x_3)}$$

$$\times e^{-ip_3(x_3-x_4)} e^{-ip_4(x_4-x_1)}$$

$$\times (-1)\,\mathrm{Tr}\left[\frac{i}{\not p_1 - M + i\varepsilon} i\gamma_5 \frac{i}{\not p_2 - M + i\varepsilon} i\gamma_5 \frac{i}{\not p_3 - M + i\varepsilon} i\gamma_5 \frac{i}{\not p_4 - M + i\varepsilon} i\gamma_5\right]$$

$$= (-ig)^4 \sqrt{\frac{1}{(2\pi)^3 2\omega_1}} \sqrt{\frac{1}{(2\pi)^4 2\omega_2}} \sqrt{\frac{1}{(2\pi)^4 2\omega_3}} \sqrt{\frac{1}{(2\pi)^4 2\omega_4}}$$

$$\times \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4}$$

$$\times (2\pi)^4 \delta(k_1 + p_1 - p_4)\,(2\pi)^4 \delta(k_2 + p_2 - p_1)$$

$$\times (2\pi)^4 \delta(p_3 - k_3 - p_2)\,(2\pi)^4 \delta(p_4 - k_4 - p_3)$$

$$\times (-1)\,\mathrm{Tr}\left[\frac{i}{\not p_1 - M + i\varepsilon} i\gamma_5 \frac{i}{\not p_2 - M + i\varepsilon} i\gamma_5 \frac{i}{\not p_3 - M + i\varepsilon} i\gamma_5 \frac{i}{\not p_4 - M + i\varepsilon} i\gamma_5\right]. \tag{4.2}$$

We can redraw Fig. 4.2 in the momentum space as

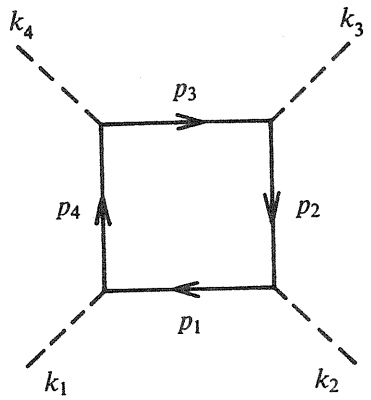


Fig. 4.3 The pion-pion amplitude of Fig. 4.1 in the momentum-space representation.

We integrate out p_2, p_3, and p_4 to give

$$A = \sqrt{\frac{1}{(2\pi)^2 2\omega_1}} \sqrt{\frac{1}{(2\pi)^3 2\omega_2}} \sqrt{\frac{1}{(2\pi)^3 2\omega_3}}$$

$$\times \sqrt{\frac{1}{(2\pi)^3 2\omega_4}} (2\pi)^4 \delta(k_1+k_2-k_3-k_4)\mathcal{M} \tag{4.3}$$

where the invariant amplitude $\mathcal{M}$ is

$$\mathcal{M} = (-ig)^4 \int \frac{d^4p_1}{(2\pi)^4} (-1) \operatorname{Tr}\left[\frac{i}{\not p_1 - M + i\varepsilon}\right.$$

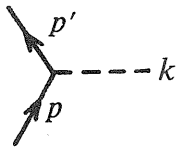
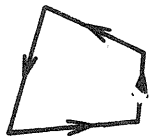
$$\times i\gamma_5 \frac{i}{\not p_1 - \not k_2 - M + i\varepsilon} i\gamma_5 \frac{i}{\not p_1 - \not k_2 + \not k_3 - M + i\varepsilon}$$

$$\left. \times i\gamma_5 \frac{i}{\not p_1 - \not k_1 - M + i\varepsilon} i\gamma_5 \right]. \tag{4.4}$$

Note that we always encounter the overall emergy-momentum conservation delta-function. The factors $\sqrt{1/(2\pi)^3 2\omega}$ are associated with the initial and the final bose particles. For Dirac particles, we have factors $\sqrt{M/(2\pi)^3 E}$ instead. In Table I, we summarize the momentum-space

Table I. Feynman rules for πN interactions.

Process	Graphical representation	amplitude
External particles	p, s $\quad p$,s	$u(p, s)$, $\bar{u}(p, s)$
	p, s $\quad p$,s	$v(p, s)$, $\bar{v}(p, s)$
	$k \quad k$	1
Internal lines	p	$iS_F(p) = \dfrac{i}{\not p - M + i\varepsilon}$
	k	$i\Delta_F(k) = \dfrac{i}{k^2 - \mu^2 + i\varepsilon}$

Vertices		$(-ig)\, i\gamma_5\, (2\pi)^4\, \delta(p-p'-k)$
Fermion loops		$(-1)\,\mathrm{Tr}\left[\quad\right]$
Internal momenta		$\int \frac{d^4p}{(2\pi)^4}$

Feynman rules for πN interactions. These rules are easier to apply than the position-space Feynman rules.

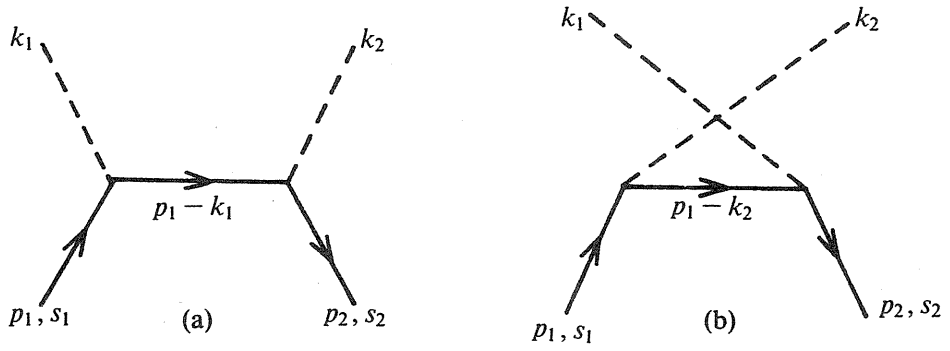


Fig. 4.4 $N\bar{N} \to \pi\pi$ amplitudes.

We shall apply these rules to obtain a few amplitudes. We look at the $N\bar{N} \to \pi\pi$ in Fig. 4.4. The invariant amplitudes are

$$\mathcal{M}_a = (-ig)^2\, \bar{v}(p_2, s_2) i\gamma_5 \frac{i}{\not{p}_1 - \not{k}_1 - M + i\varepsilon} i\gamma_5\, u(p_1, s_1) \qquad (4.5a)$$

$$\mathcal{M}_b = (-ig)^2\, \bar{v}(p_2, s_2) i\gamma_5 \frac{i}{\not{p}_1 - \not{k}_2 - M + i\varepsilon} i\gamma_5\, u(p_1, s_1)\,. \qquad (4.5b)$$

Note that we can obtain (4.5b) from (4.5a) via the interchange of k_1 and k_2:

$$\mathcal{M}_b = \mathcal{M}_a(k_1 \leftrightarrow k_2). \qquad (4.6)$$

In other words, the total amplitude $\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b$ is symmetric under the exchange of the pions, as required by the bose statistics.

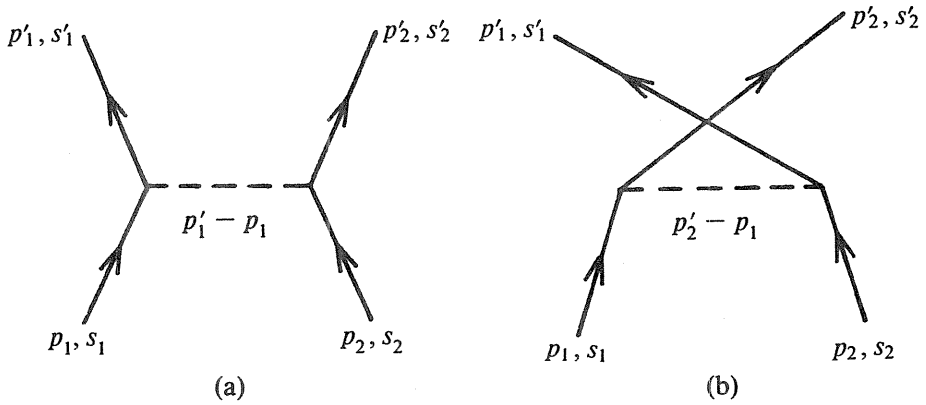


Fig. 4.5 *NN* scattering amplitude.

Next, we consider the *NN* scattering amplitudes in Fig. 4.5. We have studied the position-space amplitude for Fig. 4.5a in the last chapter. The invariant amplitudes for Figs. 4.5a and 4.5b are

$$\mathcal{M}_a = (-ig)^2\, \bar{u}(p'_1, s'_1) i\gamma_5\, u(p_1, s_1)\, \bar{u}(p'_2, s'_2) i\gamma_5\, u(p_2, s_2)$$

$$\times \frac{i}{(p'_1 - p_1)^2 - \mu^2 + i\varepsilon} \tag{4.7a}$$

$$\mathcal{M}_b = -(-ig)^2\, \bar{u}(p'_2, s'_2) i\gamma_5\, u(p_1, s_1)\, \bar{u}(p'_1, s'_1) i\gamma_5\, u(p_2, s_2)$$

$$\times \frac{i}{(p'_2 - p_1)^2 - \mu^2 + i\varepsilon}\,. \tag{4.7b}$$

Note that $\mathcal{M}_a$ and $\mathcal{M}_b$ are related by

$$\mathcal{M}_b = -\mathcal{M}_a(p'_1, s'_1 \leftrightarrow p'_2, s'_2). \tag{4.8}$$

This relation makes the total amplitude $\mathcal{M}$ antisymmetric, as required by the fermi statistics. One can verify this sign relation by a direct evaluation of the *S*-matrix.

We can generalize the above relation to relate the amplitudes for diagrams with the exchange of incoming and outgoing particles, such as those in Fig. 4.6. The amplitudes obey

$$\mathcal{M}_b = \mathcal{M}_a(k \leftrightarrow -k'). \tag{4.9}$$

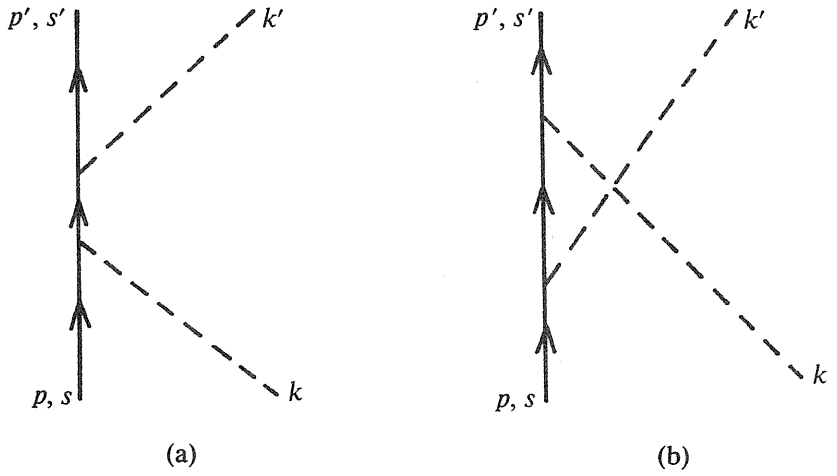


Fig. 4.6 N scattering amplitudes.

We refer to this generalized symmetry property as crossing relation. There is also a similar crossing relation for Dirac particles.

3. Cross Sections

Consider the 2-particle to n-particle process (See Fig. 4.7)

$$a + b \rightarrow 1 + 2 + \ldots + n \tag{4.10}$$

with the momenta

$$k_a + k_b = k_1 + k_2 + \ldots + k_n . \tag{4.11}$$

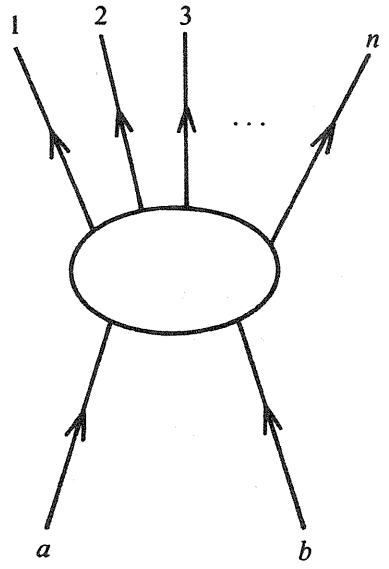


Fig. 4.7 A 2-particle to n-particle process.

The S-matrix is

$$S_{fi} = \prod_i \prod_f \sqrt{\frac{1}{V(2\omega)}}\,(2\pi)^4\delta^4(k_a+k_b-\Sigma\, k_l)\mathcal{M}, \tag{4.12}$$

where $\sqrt{1/V2\omega}$ represents the kinematical factors associated with the external bose particles, and $\mathcal{M}$ is the invariant matrix for the above process. We normalize the free particles inside a box of volume V and shall demonstrate that the cross sections do not depend on the normalization volume V. However, we shall keep the total energy-momentum conserving delta-function in (4.12) as in the continuum ($V \to \infty$) limit. Note that for a Dirac external particle, we have the kinematical factor $\sqrt{M/VE}$. However, for ease of discussion, we shall treat all kinematical factors as boson factors. Using this normalization, a one-particle state represents one particle per box of volume V.

We arbitrarily assign particle a as the incident particle, and particle b as the target. Since particle a has a number density $1/V$, the flux density of particle a relative to the target particle b is

$$\text{Flux density} = \frac{1}{V} v_{ab} \tag{4.13}$$

where $v_{ab} = |v_a - v_b|$ is the relative velocity of particle a to particle b. The transition probability of the final particles into the phase space Πd^3k_l is

$$\text{Transition Probability} = |S_{fi}|^2 \prod_l V\frac{d^3k_l}{(2\pi)^3}. \tag{4.14}$$

The cross section is defined as the transition probability per target particle, per unit time, and per unit flux. Hence we have

$$\begin{aligned} d\sigma &= \frac{\text{Transition Probability per target particle/unit time}}{\text{Flux density}} \\ &= \frac{|S_{fi}|^2 \prod_l \dfrac{Vd^3k_l}{(2\pi)^3}/T}{\dfrac{1}{V}v_{ab}} \\ &= \frac{|\mathcal{M}|^2((2\pi)^4\delta(k_a+k_b-\Sigma k_l))^2 \dfrac{1}{V2\omega_a}\dfrac{1}{V2\omega_b}\prod_l \dfrac{d^3k_l}{(2\pi)^3 2\omega_l}}{T\dfrac{1}{V}v_{ab}}, \end{aligned} \tag{4.15}$$

where T is the total time. The square of the delta-function gives

$$((2\pi)^4\delta^4(k_a+k_b-\Sigma k_l))^2 = (2\pi)^4\delta^4(k_a+k_b-\Sigma k_l)(2\pi)^4\delta^4(0). \quad (4.16)$$

It is well-known that in a finite space V and time T, we should replace $(2\pi)^4\delta^4(0)$ by VT. Hence, we have

$$((2\pi)^4\delta^4(k_a+k_b-\Sigma k_l))^2 = VT(2\pi)^4\delta^4(k_a+k_b-\Sigma k_l). \quad (4.17)$$

Then, we obtain

$$d\sigma = \frac{|\mathcal{M}|^2(2\pi)^4\delta^4\left(k_a+k_b-\sum_l k_l\right)}{2\omega_a 2\omega_b v_{ab}} \prod_m \frac{d^3k_m}{(2\pi)^3 2\omega_m}. \quad (4.18)$$

If we have one or more Dirac particles in the initial or the final state, we need to replace $1/2\omega$ by M/E in (4.18).

It is convenient to introduce the final-particle phase-space factor as

$$d\Phi_n \equiv (2\pi)^4\delta^4(P-\Sigma k_l) \prod_l \frac{d^3k_l}{(2\pi)^3 2\omega_l}. \quad (4.19)$$

where $p^\mu = (E, \mathbf{P})$ is the total energy and momentum. In terms of this phase-space factor, we can put (4.18) in a compact form,

$$d\sigma = \frac{|\mathcal{M}|^2}{2\omega_a 2\omega_b v_{ab}} d\Phi_n. \quad (4.20)$$

Using a similar method, we can work out the decay rate for the process

$$a \to 1 + 2 + \ldots n \quad (4.21)$$

with the 4-momenta

$$k_a = k_1 + k_2 + \ldots + k_n \quad (4.22)$$

The decay rate associated with the transition probability of a single particle a into a given final state Πd^3k_l is

$$d\left(\frac{1}{\tau}\right) = \frac{\text{Transition Probability into } \Pi\, d^3k_l}{\text{unit particle } a \times \text{unit time}}$$

$$
\begin{aligned}
&= |S_{fi}|^2 \prod_l \frac{V d^3k_l}{(2\pi)^3}/T \\
&= \frac{|\mathscr{M}|^2 VT(2\pi)^4 \delta^4(k_a - \Sigma k_l)}{2\omega_a VT} \prod_m \frac{d^3k_m}{(2\pi)_3 2\omega_m}. \\
&= \frac{|\mathscr{M}|^2}{2\omega_a} d\Phi_n .
\end{aligned}
\tag{4.23}
$$

4. Phase-space Factors

In computing the cross sections and the decay rates, we always encounter the same phase-space factor introduced earlier,

$$
d\Phi_n \equiv (2\pi)^4 \delta^4(\Sigma k_l - P) \prod_l \frac{d^3k_l}{(2\pi)_3 2\omega_l} \tag{4.24}
$$

where $(\omega_i, \mathbf{k}_i)$ are the final-state energy and momentum of i-th particle and $(E, \mathbf{P})$ are the total energy and momentum of the system. For 2 and 3 particle final states, we can integrate out the delta-functions to obtain some simplified formulae

$$
d\Phi_2 = \frac{k_1^3}{16\pi^2} \frac{d\Omega_1}{E k_1^2 - \omega_1 \mathbf{P}\cdot\mathbf{k}_1} \tag{4.25}
$$

$$
d\Phi_3 = \frac{k_2^2 k_1^2 dk_1 d\Omega_1 d\Omega_2}{(2\pi)^5 8\omega_1 [k_2^2 (E - \omega_1) - \omega_2 \mathbf{k}_2 \cdot (\mathbf{P} - \mathbf{k}_1)]}. \tag{4.26}
$$

Let us work (4.25) out explicitly.From (4.24), we have

$$
d\Phi_2 = (2\pi)^4 \delta^4(k_1 + k_2 - P) \frac{d^3k_1}{(2\pi)^3 2\omega_1} \frac{d^3k_2}{(2\pi)^3 2\omega_2}.
$$

We can use $\delta^3(\mathbf{k}_1 + \mathbf{k}_2 - P)$ to integrate out d^3k_2, obtaining

$$
\begin{aligned}
d\Phi_2 &= \frac{1}{(2\pi)^2} \delta(\omega_1 + \omega_2 - E) \frac{d^3k_1}{4\omega_1\omega_2} \\
&= \frac{\delta(\omega_1 + \omega_2 - E)\, k_1^2 dk_1 d\Omega_1}{16\pi^2 \omega_1 \omega_2}.
\end{aligned}
$$

Next, we use the final delta-function to integrate out dk_1. Note that

$$\omega_1 = \sqrt{k_1^2 + \mu_1^2}$$

$$\omega_2 = \sqrt{(P-k_1)^2 + \mu_2^2} = \sqrt{P^2 - 2Pk_1 \cos\theta_1 + k_1^2 + \mu_2^2}$$

are functions of k_1. We need to evaluate $\dfrac{\partial\omega_1}{\partial k_1}$ and $\dfrac{\partial\omega_2}{\partial k_1}$, obtaining

$$\frac{\partial\omega_1}{\partial k_1} = \frac{k_1}{\omega_1}, \; k_1 \equiv |\mathbf{k}_1|,$$

$$\frac{\partial\omega_2}{\partial k_1} = \frac{-P\cos\theta_1 + k_1}{\omega_2} = \frac{-\mathbf{P}\cdot\mathbf{k}_1 + k_1^2}{\omega_2 k_1}.$$

Thus, we have

$$d\Phi_2 = \frac{k_1^2 d\Omega_1}{16\pi^2\omega_1\omega_2\left(\dfrac{\partial\omega_2}{\partial k_1} + \dfrac{\partial\omega_2}{\partial k_1}\right)}$$

$$= \frac{k_1^2 d\Omega_1}{16\pi^2\omega_1\omega_2\left(\dfrac{k_1}{\omega_1} + \dfrac{-\mathbf{P}\cdot\mathbf{k}_1 + k_1^2}{\omega_2 k_1}\right)}$$

$$= \frac{k_1^3 d\Omega_1}{16\pi^2(E\,k_1^2 - \omega_1\,\mathbf{P}\cdot\mathbf{k}_1)}$$

as desired.

In particular, in the center of mass frame, we have $\mathbf{P} = 0$, $E = E_{cm}$, and

$$d\Phi_2 = \frac{k_1 d\Omega_1}{16\pi^2 E_{\text{cm}}}, \qquad \text{(c. m. frame).} \tag{4.27}$$

5. An Example

We shall work out the cross section for Fig. 4.5a. In a more realistic calculation, we should include both Figs. 4.5a and 4.5b. Since our purpose is

to demonstrate the calculation technique, we begin with the above incomplete amplitude. The invariant amplitude for Fig. 4.5a is

$$\mathcal{M} = (-ig)^2\, \bar{u}(p'_1, s'_1)\, i\gamma_5\, u(p_1, s_1)$$

$$\times\, \bar{u}(p'_2, s'_2)\, i\gamma_5\, u(p_2, s_2)\, \frac{i}{(p'_1 - p_2)^2 - \mu^2 + i\varepsilon}. \tag{4.28}$$

In the present case, the denominator of the propagator never vanishes, and the $i\varepsilon$ factor is not needed. Hence, we have

$$|\mathcal{M}|^2 = g^4\, |\bar{u}(p'_1, s'_1)\, i\gamma_5\, u(p_1, s_1)|^2$$

$$\times\, |\bar{u}(p'_2, s'_2)\, i\gamma_5\, u(p_2, s_2)|^2\, \frac{1}{[(p'_1 - p_1)^2 - \mu^2]^2}.$$

In most of the situations, we do not detect the spins of the final particles. We also use only the unpolarized initial particles. Hence, we need to sum over the final spins and average over the initial spins, giving

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4} \sum_{s_i} \sum_{s_f} |\mathcal{M}|^2$$

$$= \frac{g^4}{4} \sum_{s_i} \sum_{s_f} \bar{u}(p_1, s_1)\, i\gamma_5\, u(p'_1, s'_1)\, \bar{u}(p'_1, s'_1)\, i\gamma_5\, u(p_1, s_1)$$

$$\times\, \bar{u}(p_2, s_2)\, i\gamma_5\, u(p'_2, s'_2)\, \bar{u}(p'_2, s'_2)\, i\gamma_5\, u(p_2, s_2)$$

$$\times\, \frac{1}{[(p'_1 - p_1)^2 - \mu^2]^2}. \tag{4.29}$$

For the sum of s_1, s'_1 over the first two $\bar{u}\, i\gamma_5 u$ factors, we have

$$I \equiv \sum_{s_1, s'_1} \bar{u}(p_1, s_1)\, i\gamma_5\, u(p'_1, s'_1)\, \bar{u}(p'_1, s'_1)\, i\gamma_5\, u(p_1, s_1)$$

$$= \sum_{s_1} \bar{u}(p_1, s_1)\, i\gamma_5\, \frac{\not{p}_1 + M}{2M}\, i\gamma_5\, u(p_1, s_1) \tag{4.30}$$

In (4.30), we have used the identity

$$\sum_{s'_1} u(p'_1, s'_1)\,\bar{u}(p'_1, s'_1) = \frac{\not{p}_1 + M}{2M}.$$

To continue the simplification, we need to move $\bar{u}(p_1, s_1)$ from the begining of the expression to the end so that we can perform a sum of s_1 over $\bar{u}(p_1, s_1)\,\bar{u}(p_1, s_1)$. We can accomplish this by treating the scalar expression (4.30) as a trace of a 1×1 matrix, and then move the 1×4 matrix $\bar{u}(p_1, s_1)$ under the trace to the end, giving

$$\begin{aligned} I &= \sum_{s_1} \mathrm{Tr}\left[\bar{u}(p_1, s_1)\, i\gamma_5 \frac{\not{p}'_1 + M}{2M} i\gamma_5\, u(p_1, s_1)\right] \\ &= \sum_{s_1} \mathrm{Tr}\left[i\gamma_5 \frac{\not{p}'_1 + M}{2M} i\gamma_5\, u(p_1, s_1)\,\bar{u}(p_1, s_1)\right] \qquad (4.31) \\ &= \mathrm{Tr}\left[i\gamma_5 \frac{\not{p}'_1 + M}{2M} i\gamma_5 \frac{\not{p}_1 + M}{2M}\right]. \end{aligned}$$

We can apply the same technique to the sum over the next two $\bar{u}i\gamma_5 u$ factors. Combining these two trace factors, we have

$$\begin{aligned} \overline{|\mathscr{M}^2|} &= \frac{g^2}{4} \mathrm{Tr}\left(i\gamma_5 \frac{\not{p}'_1 + M}{2M} i\gamma_5 \frac{\not{p}_1 + M}{2M}\right). \\ &\quad \times \mathrm{Tr}\left(i\gamma_5 \frac{\not{p}'_1 + M}{2M} i\gamma_5 \frac{\not{p}_2 + M}{2M}\right). \qquad (4.32) \\ &\quad \times \frac{1}{((p'_1 - p_1)^2 - \mu^2)^2}. \end{aligned}$$

We need a few identities for evaluating the traces over the γ-matrices. They are

$$\text{Tr}\,(1) = 4$$

$$\text{Tr}\,(\text{odd numbers of } \gamma_\mu\text{'s}) = 0,$$

$$\text{Tr}\,(\gamma_5) = \text{Tr}\,(\gamma_\mu\gamma_5) = \text{Tr}\,(\gamma_\mu\gamma_\nu\gamma_5)$$

$$= \text{Tr}\,(\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_5) = 0 \tag{4.33}$$

$$\text{Tr}\,(\not a\not b\not c\not d) = 4\,(a\cdot b\, c\cdot d - a\cdot c\, b\cdot d + a\cdot d\, c\cdot b),$$

$$\text{Tr}\,(\gamma_5\not a\not b\not c\not d) = 4i\varepsilon_{\alpha\beta\gamma\delta}\, a_\alpha b_\beta c_\gamma d_\delta.$$

In addition to the above trace identities, we also find the following identities of γ-matrices useful:

$$\gamma_\mu \not a\, \gamma_\mu = -2\not a$$

$$\gamma_\mu \not a\not b\, \gamma_\mu = 4\, a\cdot b \tag{4.34}$$

$$\gamma_\mu \not a\not b\not c\, \gamma_\mu = -2\not c\not b\not a.$$

Now, we are ready for the reduction. The first trace gives

$$\text{Tr}\left[i\gamma_5 \frac{\not p_1' + M}{2M} i\gamma_5 \frac{\not p_1 + M}{2M}\right]$$

$$= -\text{Tr}\,[(M - \not p_1')(\not p_1 + M)]/(4M^2)$$

$$= (\not p_1'\cdot p_1 - M^2)/M^2.$$

Similarly, we have for the second trace

$$\text{Tr}\left[i\gamma_5 \frac{\not p_2' + M}{2M} i\gamma_5 \frac{\not p_2 + M}{2M}\right] = (p_2\cdot p_2' - M'^2)/M^2.$$

Putting these results together, we have

$$\overline{|\mathcal{M}^2|} = \frac{g^4}{4M^4} \frac{(p_1\cdot p_1' - M^2)(p_2\cdot p_2' - M^2)}{((p_1' - p_1)^2 - \mu^2)^2}. \tag{4.35}$$

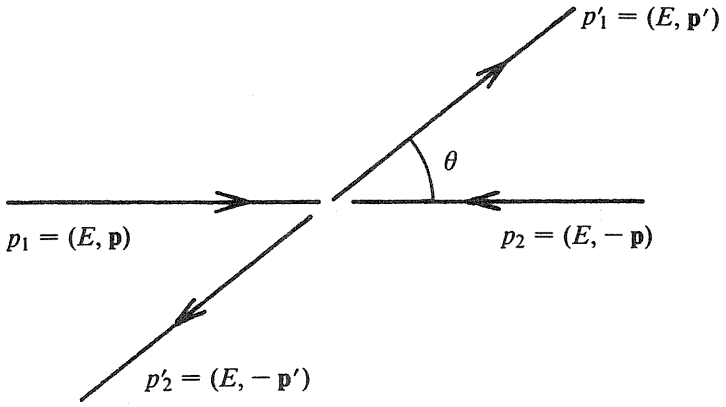


Fig. 4.8 *NN* center of mass frame.

In the center of mass frame (see Fig. 4.8), we have

$$p_1 = (E, \mathbf{p}), p_2 = (E, -\mathbf{p}), E = \sqrt{p^2+M^2}$$

$$p'_1 = (E, \mathbf{p}'), p'_2 = (E, -\mathbf{p}'), |\mathbf{p}| = |\mathbf{p}'| = p,$$

$$p_1 \cdot p'_1 - M^2 = E^2 - \mathbf{p}\cdot\mathbf{p}' - M^2 = p^2(1 - \cos\theta) \tag{4.36}$$

$$p_2 \cdot p'_2 - M^2 = E^2 - \mathbf{p}\cdot\mathbf{p}' - M^2 = p^2(1 - \cos\theta)$$

$$(p'_1 - p_1)^2 = -(\mathbf{p}'_1 - \mathbf{p}_1)^2 = -2p^2(1 - \cos\theta).$$

Hence, we have

$$\overline{|\mathscr{M}^2|} = \frac{g^4}{4M^4} \frac{(p^2(1 - \cos\theta))^2}{[2p^2(1 - \cos\theta) + \mu^2]^2} . \tag{4.37}$$

Since the external particles are Dirac particles, we need to modify the phase space factor to

$$d\Phi_2 = (2M)^4 \cdot \frac{p_1 \, d\Omega_1}{16\pi^2(2E)}. \tag{4.38}$$

Note that the center of mass energy is $2E$. Hence, we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E\ 2E\ 2p/E} \overline{|\mathcal{M}|}^2 (2M)^4 \frac{p}{32\pi^2 E}$$

$$= \frac{g^4}{4M^4} \cdot \frac{(p^2(1-\cos\theta))^2}{(2p^2(1-\cos\theta)+\mu^2)^2} \cdot \frac{(2M)^4}{8E^2 \times 32\pi^2} \tag{4.39}$$

$$= \frac{g^4}{64\pi^2 E^2} \frac{(p^2(1-\cos\theta))^2}{(2p^2(1-\cos\theta)+\mu^2)^2}.$$

It is easy to verify that $\frac{d\sigma}{d\Omega}$ has the dimension of area.

Remark: One way to take care of the extra $2M$-factors associated with the initial and the final Dirac particles is to introduce a different normalization condition for u, $\bar{u}$, v, and $\bar{v}$. The required normalization condition is

$$\bar{u}(p, s)\, u(p, s') = \delta_{ss'}\, 2M,$$

$$\bar{u}(p, s)\, v(p, s') = \bar{v}(p, s)\, u(p, s') = 0$$

$$\bar{v}(p, s)\, v(p, s') = -\delta_{ss'}\, 2M\,. \tag{4.40}$$

This normalization condition leads to

$$\sum_s u(p, s)\, \bar{u}(p, s) = \not{p} + M$$

$$-\sum_s v(p, s)\, \bar{v}(p, s) = M - \not{p}. \tag{4.41}$$

Both the invariant amplitude and the phase space-factor are finite as $M \to 0$. This normalization is very convenient for describing massless fermions such as neutrinos.

Chapter 5

QUANTUM ELECTRODYNAMICS

1. The Classical Electromagnetic Field

We may describe the electromagnetic fields (**E**, **B**) by an antisymmetric tensor $F_{\mu\nu}$ with

$$E^i = F^{0i} \tag{5.1}$$

$$B^i = \frac{1}{2}\varepsilon^{ijk} F_{jk} . \tag{5.2}$$

The Maxwell equations are

$$\partial_\mu F_{\mu\nu} = j_\nu \tag{5.3}$$

and

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{5.4}$$

where $j^\mu = (\rho, \mathbf{j})$ are the external charge and current densities. We can satisfy (5.4) identically if we introduce a vector potential A_μ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{5.5}$$

In terms of A_μ, Eq. (5.3) becomes

$$\partial^2 A_\nu - \partial_\mu \partial_\nu A_\mu = j_\nu . \tag{5.6}$$

Equation (5.6) does not determine A_μ uniquely. Indeed, if we change A_μ by a gradient

$$A_\mu \to A_\mu + \partial_\mu \lambda , \tag{5.7}$$

then, both $F_{\mu\nu}$ and Eq. (5.6) are invariant under this transofrmation

$$F_{\mu\nu} \to F_{\mu\nu}\,. \tag{5.8}$$

We call such a transformation a gauge transformation.

First, we wish to study A_μ and $F_{\mu\nu}$ as a classical system with a given external current j_μ. Obviously, not all components of A_μ are dynamical variables. We can impose some conditions on A_μ by a proper choice of the gauge parameter λ.

We start with the Coulomb gauge defined by

$$\nabla\cdot\mathbf{A} = 0. \tag{5.9}$$

Under the gauge condition (5.9), the $\nu = 0$ component of Eq. (5.6) becomes

$$(\partial_0^2 - \nabla^2)\,A_0 - \partial_0\,(\partial_0 A_0 - \nabla\cdot A) = j_0$$

or

$$\nabla^2 A_0 = j_0. \tag{5.10}$$

Equation (5.10) is the well-known Poisson's equation. With an appropriate boundary condition, we can integrate (5.10) to give

$$A_0\,(x,\,t) = \int d^3y \frac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|} j_0\,(y,\,t). \tag{5.11}$$

Hence, in the Coulomb gauge, the scalar potential A_0 is determined by the Coulomb law. Thus, there are only two independent field variables. In the Coulomb gauge, the independent fields are $(A^\perp)_k$.

We examine the spatial components of (5.6) in more detail

$$(\partial_0^2 - \nabla^2)\,A_k - \partial_k\,\partial_0\,A_0 = j_k\,. \tag{5.12}$$

To check the consistency of our gauge condition, we differentiate (5.12) with respect to x_k, giving

$$-\nabla^2\partial_0 A_0 = \partial_k j_k.$$

Using (5.10) to eliminate $\nabla^2 A_0$, we obtain a constraint on j_μ as

$$\partial_0 j_0 - \partial_k j_k \equiv \partial_\mu j_\mu = 0. \tag{5.13}$$

This is the well-known current conservation condition, which follows directly from Eq. (5.3).

Given that the longitudinal component of (5.12) is satisfied, we obtain the transverse equation as

$$(\partial_0^2 - \nabla^2) A_k = j_k - \frac{\partial_k \partial_l}{\nabla^2} j_l. \tag{5.14}$$

In Eq. (5.14), $-1/\nabla^2$ is the shorthand notation for the integral over the Coulomb Green's function $1/4\pi \left| \mathbf{x} - \mathbf{x}' \right|$.

We can reproduce the Maxwell equation from a Lagrange function

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - A_\mu j_\mu \tag{5.15}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{5.16}$$

In the absence of any external currents, we have the Lagrange function as

$$\mathscr{L} = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) \tag{5.17}$$

and the Hamiltonian density as

$$\mathscr{H} = \frac{1}{2} (\mathbf{E}^2 + \mathbf{B}^2) . \tag{5.18}$$

Now, we wish to construct the properly normalized plane wave solutions. Of course, these solutions depend on the gauge of our choice. From (5.14), we learn that in the Coulomb gauge the nontrivial components, $A_k^\perp$, obey the Klein-Gordon equation. Thus we begin with the exponential factor exp $(\pm ik \cdot x)$ as before. We also note that A_μ is a 4-component object. Hence, in analog to Dirac spinors u, $\bar{u}$, v, and $\bar{v}$, we also include the polarization vectors $\varepsilon_\mu (k, \lambda)$ in the expansion, giving

$$A_\mu(x, t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \sum_{\lambda=1}^{2} \varepsilon_\mu(k, \lambda) \left[a(k, \lambda)\, e^{-ik\cdot x} + a^\dagger(k, \lambda)\, e^{ik\cdot x}\right] \quad (5.19)$$

where λ describes the polarization of the plane wave solution. Since A_μ is real, we choose ε_μ to be real. Coefficients $a(k, \lambda)$ and $a^\dagger(k, \lambda)$ are complex conjugations of each other.

In order to satisfy the Coulomb gauge condition,

$$\nabla \cdot \mathbf{A} = 0, \qquad A_0 = 0, \quad (5.20)$$

we impose the conditions,

$$\mathbf{k}\cdot\boldsymbol{\varepsilon}(k, \lambda) = 0, \quad \varepsilon_0 = 0. \quad (5.21)$$

We also choose the normalization conditions of $\varepsilon(k, \lambda)$ as

$$\begin{aligned} \boldsymbol{\varepsilon}(k, \lambda) \cdot \boldsymbol{\varepsilon}(k, \lambda') &= -\varepsilon_\mu(k, \lambda)\, \varepsilon_\mu(k, \lambda') \\ &= \delta_{\lambda\lambda'}, \qquad \lambda, \lambda' = 1, 2. \end{aligned} \quad (5.22)$$

We can define a spin projection operator in the Coulomb gauge by

$$\Lambda_{\mu\nu} = \sum_\lambda \varepsilon_\mu(k, \lambda)\, \varepsilon_\nu(k, \lambda). \quad (5.23)$$

Since

$$\varepsilon_0 = 0, \qquad \mathbf{k}\cdot\boldsymbol{\varepsilon} = 0,$$

and

$$\boldsymbol{\varepsilon}(k, \lambda)\cdot\boldsymbol{\varepsilon}(k, \lambda') = \delta_{\lambda\lambda'},$$

we have

$$\Lambda_{00} = 0 \quad (5.24a)$$

$$\Lambda_{0l} = 0 \quad (5.24b)$$

$$\Lambda_{lm} = \sum_\lambda \varepsilon_l(k, \lambda)\, \varepsilon_m(k, \lambda) = \delta_{lm} - \frac{k_l k_m}{\mathbf{k}^2}. \quad (5.24c)$$

In term of special 4-vectors,

$$\eta_\mu = (1, 0, 0, 0) \tag{5.25}$$

$$\bar{k}_\mu = (0, \mathbf{k})_\mu = k_\mu - (\mathbf{k}\cdot\boldsymbol{\eta})\,\eta_\mu \tag{5.26}$$

$$\bar{k}^2 = -\,\mathbf{k}^2 \tag{5.27}$$

we have

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \frac{\bar{k}_\mu k_\nu + \bar{k}_\nu k_\mu - k_\mu k_\nu + k^2\eta_\mu\eta_\nu}{\bar{k}^2}. \tag{5.28}$$

It is straightforward to verify that (5.28) is equivalent to (5.24).

2. Quantization of the Electromagentic Field

From the Lagrange function $\mathscr{L}$, we can identify the momentum conjugate to A_k as the E-field,

$$\Pi_k = \frac{\partial\mathscr{L}}{\partial\dot{A}_k} = F_{0k} = E_k\,. \tag{5.29}$$

In the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$, only $A_k^\perp$ are the dynamical variables. We expect that only $E_k^\perp$ are their conjugate variables. Indeed, we find that the longitudinal component of $\mathbf{E}$ obeys

$$\nabla\cdot\mathbf{E} = j_0 \tag{5.30}$$

which is obviously not a conjugate variable. The equal-time canonical quantization relations give

$$[A_k^\perp(x, t), A_l^\perp(y, t)] = 0, \tag{5.31a}$$

$$[E_k^\perp(y, t), E_l^\perp(y, t)] = 0, \tag{5.31b}$$

$$[A_k^\perp(x, t), E_l^\perp(y, t)] = i\left(\delta_{kl} - \frac{\partial_k\partial_l}{\nabla^2}\right)\delta(x-y)$$

$$= i\left[\delta_{kl}\,\delta(x-y) + \partial_k\partial_l\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}\right]. \tag{5.31c}$$

Note that these quantization rules are gauge dependent, and nonlocal. However, it is pleasing to know that the quantization rules among the E-fields and the B-fields are both gauge invariant and local. We can work out their quantization rules straightforwardly as

$$[E_k(x,t), E_l(y,t)] = [B_k(x,t), B_l(y,t)] = 0 \tag{5.32a}$$

$$\begin{aligned}[E_k(x,t), B_l(y,t)] &= \left[E_k(x,t), \varepsilon_{lmn}\left(\frac{\partial}{\partial y}\right)_m A_n(y,t)\right] \\ &= i\varepsilon_{lmn}\left(\frac{\partial}{\partial x}\right)_m \left(\delta_{nk} - \frac{\partial_n \partial_k}{\nabla^2}\right)\delta(x-y) \\ &= i\varepsilon_{klm}\partial_m \delta(x-y). \end{aligned} \tag{5.32b}$$

Knowing the quantization rules, we can interpret Eq. (5.19) as the expansion of a quantum field in terms of the plane wave solutions. The expansion coefficients $a(k,\lambda)$, $a^\dagger(k,\lambda)$ are time and space independent operators. It is straightforward to verify that a and $a^\dagger$ obey

$$[a(k,\lambda), a(k',\lambda')] = [a^\dagger(k,\lambda), a^\dagger(k',\lambda')] = 0 \tag{5.33a}$$

$$[a(k,\lambda), a^\dagger(k',\lambda')] = \delta^3(k-k')\,\delta_{\lambda\lambda'}. \tag{5.33b}$$

Thus, we may interpret a and $a^\dagger$ as the annihilation and the creation operators. From the Hamiltonian density $\mathscr{H}$, we can work out the Hamiltonian operator as

$$\begin{aligned} \mathrm{H} &= \int d^3x\, \mathscr{H} = \sum_\lambda \int d^3k \frac{\omega_k}{2}\left(a^\dagger(k,\lambda)\, a(k,\lambda) + a(k,\lambda)\, a^\dagger(k,\lambda)\right) \\ &= \sum_\lambda \int d^3k\, \omega_k \left(a^\dagger(k,\lambda)\, a(k,\lambda) + \tfrac{1}{2}\right). \end{aligned} \tag{5.34}$$

with

$$\omega_k \equiv |k|.$$

Just as in the scalar case, we can introduce a new Hamiltonian operator with the infinite constant

$$\sum_{\lambda} \int d^3k\, \omega_k/2$$

removed. This new Hamitonian is the same as (5.34) except that it is normal-ordered.

Similarly, we can introduce a momentum operator as

$$\mathbf{P} = \int d^3x : \mathbf{E} \times \mathbf{B} :$$

$$= \sum_{\lambda} \int d^3k\, \mathbf{k}\, a^{\dagger}(k, \lambda)\, a(k, \lambda). \tag{5.35}$$

We may introduce the ground state $|0\rangle$, the one-photon states, and the multiphoton state as

$$a(k, \lambda)\,|0|\rangle = 0, \qquad \text{all } k, \lambda \tag{5.36}$$

$$|k, \lambda\rangle = a^{\dagger}(k, \lambda)|0\rangle, \ldots, \text{etc.} \tag{5.37}$$

3. The Photon Propagators

We define the photon propagation function in the Coulomb gauge by

$$T(A_{\mu}(x)\, A_{\nu}(y)) = :A_{\mu}(x)\, A_{\nu}(y): + \overbrace{A_{\mu}(x)\, A_{\nu}(y)}. \tag{5.38}$$

The contraction gives

$$\overbrace{A_{\mu}(x)\, A_{\nu}(y)} = \int \frac{d^3k}{2\omega_k(2\pi)^3} \sum_{\lambda} \varepsilon_{\mu}(k, \lambda)\, \varepsilon_{\nu}(k, \lambda)$$

$$\times \left[\theta(x_0 - y_0)\, e^{-ik\cdot(x-y)} + \theta(y_0 - x_0)\, e^{-ik\cdot(y-x)}\right] \tag{5.39}$$

where $k_0 = \omega = |k|$ in (5.39) is the energy of a real photon. We can express (5.39) as a 4-dimensional expansion,

$$\overbrace{A_{\mu}(x)\, A_{\nu}(y)} = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x-y)}\, \Lambda_{\mu\nu}(k)\, \frac{i}{k^2 + i\varepsilon} \tag{5.40}$$

where k_0 is now an integration variable, and $\Lambda_{\mu\nu}$ is (See Eq. (5.28))

$$\Lambda_{\mu\nu} = -g_{\mu\nu} + \frac{\bar{k}_\mu k_\nu + \bar{k}_\nu k_\mu - k_\mu k_\nu}{\bar{k}^2}$$

$$+ \frac{k^2}{\bar{k}^2}\eta_\mu\eta_\nu \,. \tag{5.41}$$

The last term of $\Lambda_{\mu\nu}$ leads to a contribution to $\overline{A_\mu(x)\,A_\nu(y)}$ as

$$\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{i}{k^2+i\varepsilon} \frac{k^2}{-\mathbf{k}^2} \eta_\mu\eta_\nu$$

$$= \delta(x_0-y_0)\,\eta_\mu\eta_\nu \int \frac{d^3k}{(2\pi)^3} i\frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{-\mathbf{k}^2}$$

$$= \delta(x_0-y_0)\,\eta_\mu\eta_\nu \frac{-i}{4\pi\,|\mathbf{x}-\mathbf{y}|}\,. \tag{5.42}$$

This term has the same structure of a Coulomb interaction, but has the opposite sign. It cancels the explicit Coulomb interaction appearing in the Coublomb gauge calculation.

To see the presence of the Coulomb term in the Hamiltonian, we begin with a Lagrange function of the Maxwell field in the presence of an external current j_μ,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - A_\mu j_\mu$$

$$= \frac{1}{2}(F_{0k})^2 - \frac{1}{4}F_{kl}^2 - A_\mu j_\mu \,. \tag{5.43}$$

In the Coulomb gauge, we have $\nabla\cdot\mathbf{A} = 0$, and

$$F_{0k} = \partial_0 A_k - \partial_k A_0 = E_k^\perp - \partial_k A_0, \tag{5.44}$$

and hence

$$\mathscr{L} = \frac{1}{2}(\partial_0 A_k)^2 + \frac{1}{2}(\partial_k A_0)^2 - \frac{1}{4}F_{kl}^2 - A_\mu j_\mu - \partial_k(\cancel{\partial_0 A_k A_0}). \tag{5.45}$$

The last term in (5.45) will not contribute because it is a total derivative.

We shall ignore it. From (5.45), we obtain the Hamiltonian density as

$$\mathscr{H} = \frac{\partial \mathscr{L}}{\partial \dot{A}_k} \dot{A}_k - \mathscr{L}$$

$$= \frac{1}{2}(\partial_0 A_k)^2 + \frac{1}{4} F_{kl}^2 - \frac{1}{2}(\partial_k A_0)^2 + A_\mu j_\mu$$

$$= \frac{1}{2}((\mathbf{E}^\perp)^2 + \mathbf{B}^2) + \frac{1}{2} A_0 \nabla^2 A_0 + A_0 j_0 - A_k j_k$$

$$- \frac{1}{2} \partial_k (\cancel{\partial_k A_0 A_0}). \tag{5.46}$$

Again, we shall ignore the last term in (5.46) because it is also a total derivative. After integration and with the use of Eq. (5.11), we obtain

$$H = H_0 + \frac{1}{2} \int d^3x d^3y \, j_0(x) \frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} j_0(y) - \int d^3x \, A_k j_k \,. \tag{5.47}$$

The second term in (5.47) is the explicit Coulomb term which cancels the $\eta_\mu \eta_\nu$ term in (5.42).

After cancelling out the Coulomb term, we can express our photon propagator as

$$iD_{F\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} + \text{terms containing } k_\mu \text{ or } k_\nu \,. \tag{5.48}$$

Because of the current conservation, the k_μ and k_ν terms do not contribute to any scattering amplitude. It is sufficient to keep the $g_{\mu\nu}$ term only in (5.48).

4. The Minimal Coupling and the Feynman Rules

It appears that the nature honors the gauge invariance even in the presence of interactions. We can guarantee this invariance property if we construct the interaction between the electromagnetic field and a charged field by the minimal substitution of the derivatives on the charged field,

$$\partial_\mu \to \partial_\mu + ie A_\mu \,. \tag{5.49}$$

The Lagrange function for the interacting electrons and the electromagnetic field, known as the quantum electrodynamics (QED), is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} \left[i\gamma_\mu (\partial_\mu + ie A_\mu) - m \right] \psi \tag{5.50}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{5.51}$$

In (5.50), m and e are the mass and the electric charge of the electron. The field equations are

$$\partial_\mu F_{\mu\nu} = e\bar{\psi} \gamma_\nu \psi \equiv j_\nu \tag{5.52}$$

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\, \psi = 0 \tag{5.53}$$

$$\bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - eA\!\!\!/ - m) = 0. \tag{5.54}$$

It is easy to show that (5.53) and (5.54) imply the current conservation,

$$\partial_\mu j_\mu = 0 \tag{5.55}$$

which is required for consistency.

Let us investigate the question of gauge transformations. It is straightforward to see that we need to transform both A_μ and ψ in order to preserve field equations (5.53) and (5.54). The required transformation is

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \lambda(x), \tag{5.56}$$

$$\psi(x) \to \psi'(x) = e^{-ie\lambda(x)} \psi(x) \tag{5.57a}$$

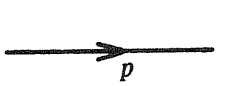
$$\psi^\dagger(x) \to \psi^\dagger(x') = \psi^\dagger(x)\, e^{ie\lambda(x)}. \tag{5.57b}$$

The Dirac field is subject to a coordinate dependent phase transformation. Note that $\partial_\mu \psi$ does not have a simple transformation law. The combination $(\partial_\mu + ieA_\mu)\psi$ does:

$$\begin{aligned} (\partial_\mu + ieA_\mu)\, \psi &\to (\partial_\mu + ieA'_\mu)\, \psi' \\ &= (\partial_\mu + ieA_\mu + ie\partial_\mu \lambda)\, e^{-ie\lambda(x)} \psi \\ &= e^{-ie\lambda(x)} (\partial_\mu + ieA_\mu)\psi. \end{aligned} \tag{5.58}$$

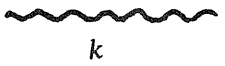
It is easy to verify that both the Lagrange function and the field equations (5.51)–(5.54) are invariant under the gauge trnasformation (5.56)–(5.57).

Using the gauge freedom, we can choose a particular gauge such as the Coulomb gauge, and quantize the A_μ and ψ fields accordingly. Their canonical quantization rules are the same as those of the free fields. We can follow the Dyson-Wick expansion and obtain a set of Feynman rules. In addition to the electron propagators, we also encounter the photon propagators and the photon-electron vertices.



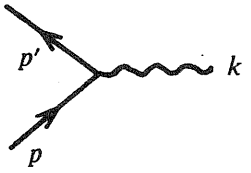
$$iS_F(p) = \frac{i}{\not{p} - m + i\varepsilon}$$

(a)



$$iD_F(k)_{\mu\nu} = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}$$

(b)



$$(-ie)\gamma_\mu (2\pi)^4 \delta(p - p' - k)$$

(c)

Fig. 5.1 Feynman rules associated with (a) an electron propagator, (b) a photon propagator, and (c) a photon-electron vertex.

In Fig. 5.1, we denote the Feynman diagrams and their corresponding amplitudes. As we have mentioned earlier, the photon propagator may contain other pieces which have either a k_μ or a k_ν. These additional terms do not contribute to the scattering amplitude, and we have ignored them. Using the path integral technique, we can derive the above simplified photon propagator directly. We shall discuss this powerful technique in the Advanced Field Theory.

For the external particles, we have the Feynman graphs and their amplitudes as shown in Fig. 5.2. The other rules are identical to those of the pion-nucleon interaction theory.

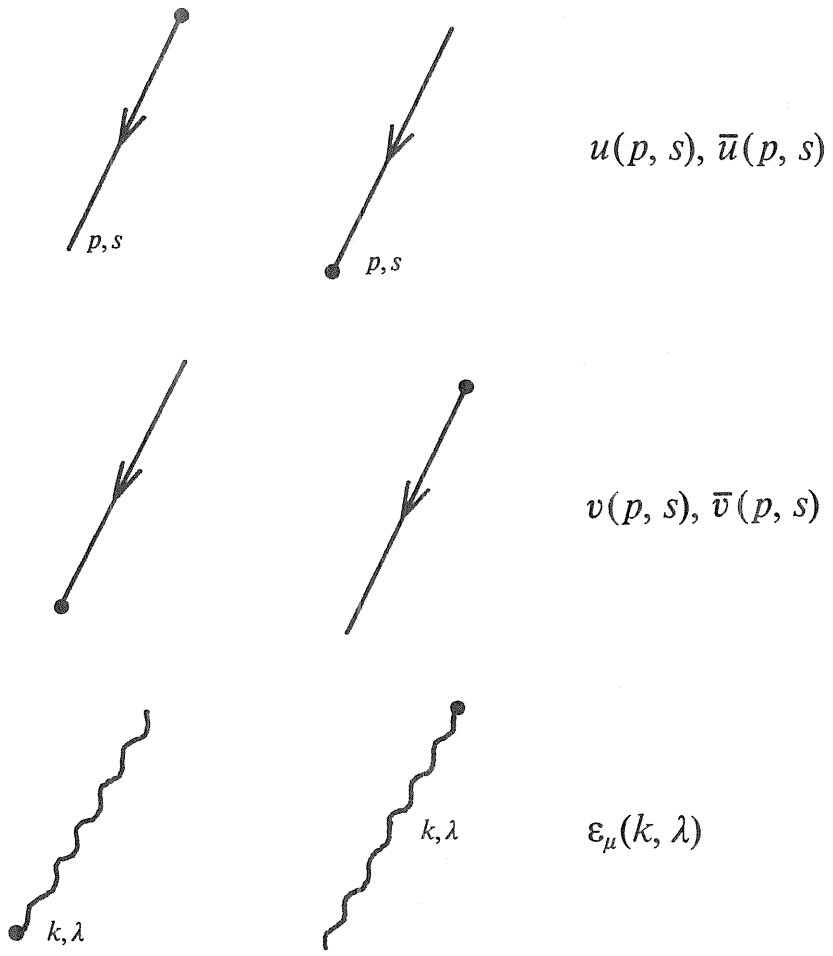


Fig. 5.2 External particles in QED.

5. Electron-Electron Scattering

Just as in the nucleon-nucleon scattering, the second order electron-electron scattering contains two graphs:

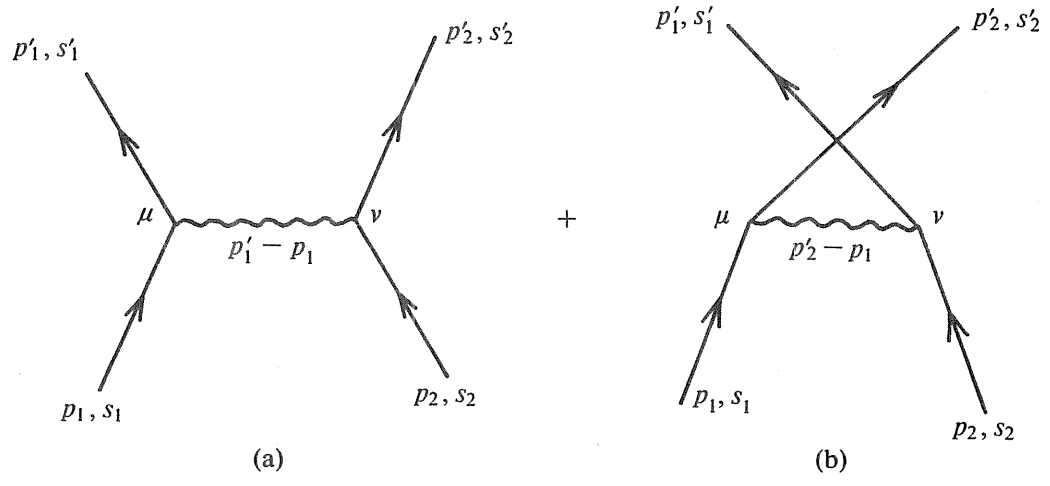


Fig. 5.3 Electron-electron scattering.

The invariant amplitudes of these diagrams are

$$\mathcal{M}_a = \bar{u}(p'_1, s'_1)(-ie\gamma_\mu)\, u(p_1, s_1)\bar{u}(p'_2, s'_2)(-ie\gamma_\nu)\, u(p_2, s_2)$$

$$\times \frac{-ig_{\mu\nu}}{(p'_1 - p_1)^2 + i\varepsilon} \tag{5.59a}$$

$$\mathcal{M}_b = -\bar{u}(p'_2, s'_2)(-ie\gamma_\mu)\, u(p_1, s_1)\bar{u}(p'_1, s'_1)(-ie\gamma_\nu)\, u(p_2, s_2)$$

$$\times \frac{-ig_{\mu\nu}}{(p'_2 - p'_1)^2 + i\varepsilon} \tag{5.59b}$$

and

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b\,. \tag{5.60}$$

In computing the differential cross section, we need to evaluate

$$|\mathcal{M}|^2 = |\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 + 2\,\mathrm{Re}\,\mathcal{M}_a\mathcal{M}_b^*\,. \tag{5.61}$$

We shall evaluate each term separately. We assume that we do not detect the final electron spins, and use the unpolarized electron beam and target. We have

$$|\mathcal{M}_a|^2 = \frac{1}{4}\sum_{s_i}\sum_{s_f}|\mathcal{M}_a|^2$$

$$= \frac{e^4}{4}\sum_{s_i,s_f}\bar{u}(p'_1, s'_1)\,\gamma_\mu\, u(p_1, s_1)\,\bar{u}(p'_2, s'_2)\,\gamma_\mu\, u(p_2, s_2)$$

$$\times\, \bar{u}(p_1, s_1)\,\gamma_\nu\, u(p'_1, s'_1)\,\bar{u}(p_2, s_2)\,\gamma_\nu\, u(p'_2, s'_2)$$

$$\times \frac{1}{[(p'_1 - p_1)^2]^2}$$

$$= \frac{e^4}{4}\sum_{s_i,s_f}\bar{u}(p_1, s_1)\,\gamma_\nu\, u(p'_1, s'_1)\,\bar{u}(p'_1, s'_1)\,\gamma_\mu\, u(p_1, s_1)$$

$$\times\, \bar{u}(p_2, s_2)\,\gamma_\nu\, u(p'_2, s'_2)\,\bar{u}(p'_2, s'_2)\,\gamma_\mu\, u(p_2, s_2)$$

$$\times \frac{1}{[(p'_1-p'_1)^2]^2}$$

$$= \frac{e^4}{4} \operatorname{Tr}\left(\gamma_\nu \frac{\not{p}'_1+m}{2m} \gamma_\mu \frac{\not{p}_1+m}{2m}\right) \operatorname{Tr}\left(\gamma_\nu \frac{\not{p}'_2+m}{2m} \gamma_\mu \frac{\not{p}_2+m}{2m}\right)$$

$$\times \frac{1}{[(p'_1-p_1)^2]^2} \tag{5.62a}$$

$$\overline{|\mathcal{M}_b|^2} = \overline{|\mathcal{M}_a|^2}\,(p'_1 \longleftrightarrow p'_2) \tag{5.62b}$$

and

$$\overline{\operatorname{Re}(\mathcal{M}_a \mathcal{M}_b^*)} = -\frac{e^4}{4} \sum_{s_i} \sum_{s_f} \frac{1}{(p'_1-p_1)^2\,(p'_2-p_1)^2}$$

$$\times \operatorname{Re}\left[\bar{u}(p'_1, s'_1)\,\gamma_\mu\, u(p_1, s_1)\,\bar{u}(p'_2, s'_2)\,\gamma_\mu\, u(p_2, s_2)\right.$$

$$\left.\times\, \bar{u}(p_1, s_1)\,\gamma_\nu\, u(p'_2, s'_2)\,\bar{u}(p_2, s_2)\,\gamma_\nu\, u(p'_1, s'_1)\right] \tag{5.62c}$$

$$= -\frac{e^4}{4} \frac{1}{(p'_1-p_1)^2\,(p'_2-p_1)^2}$$

$$\times \operatorname{Tr}\left[\gamma_\nu \frac{\not{p}'_1+m}{2m} \gamma_\mu \frac{\not{p}_1+m}{2m} \gamma_\nu \frac{\not{p}'_2+m}{2m} \gamma_\mu \frac{\not{p}_2+m}{2m}\right].$$

We drop the Re in the last expression because it is real by itself. The rest of the calculation is just pure algebra, and we have

$$\operatorname{Tr}\left[\gamma_\nu(\not{p}'_1+m)\,\gamma_\mu(\not{p}_1+m)\right]$$

$$= 4\left(g_{\mu\nu}\,m^2 + p_{1_\mu} p'_{1_\nu} + p_{1_\nu} p'_{1_\mu} - g_{\mu\nu}\, p'_1 \cdot p_1\right), \tag{5.63a}$$

$$\gamma_\nu(\not{p}'_1+m)\,\gamma_\mu(\not{p}_1+m)\,\gamma_\nu$$

$$= -2\,\not{p}_1\,\gamma_\mu\,\not{p}'_1 + 4m\left(p_{1_\mu} + p'_{1_\mu}\right) - 2m^2\,\gamma_\mu, \tag{5.63b}$$

$$\begin{aligned}
&\mathrm{Tr}\,[\gamma_\nu(\not p'_1+m)\,\gamma_\mu(\not p_1+m)\,\gamma_\nu(\not p'_2+m)\,\gamma_\mu(\not p_2+m)] \\
&\quad=\mathrm{Tr}\,[(-2\not p_1\,\gamma_\mu\not p'_1+4m(p_1+p'_1)_\mu-2m^2\gamma_\mu) \\
&\qquad\times(\not p'_2+m)\,\gamma_\mu(\not p_2+m)] \\
&\quad=\mathrm{Tr}\,[-2\not p_1(4p'_1\cdot p'_2-2m\not p'_1)(\not p_2+m) \\
&\qquad+4m(\not p'_2+m)(\not p_1+\not p'_1)(\not p_2+m)-2m^2(-2\not p'_2+4m)(\not p_2+m)] \\
&\quad=-32\,p_1\cdot p_2\,p'_1\cdot p'_2+16m^2\,p_1\cdot p'_1+16m^2\,(p_1+p'_1)\cdot(p_2+p'_2) \\
&\qquad+16m^2\,p_2\cdot p'_2-32m^4. \qquad (5.63c)
\end{aligned}$$

When we put all these trace expressions into (5.61), we obtain the cross section in the c.m. frame as (called Møller's formula)

$$\frac{d\sigma}{d\Omega}=\frac{1}{2E\;2E\;2p/E}\overline{|\mathscr{M}^2|}\,(2m)^4\frac{p}{16\pi^2(2E)}.$$

The cross section in the limit of $E>>m$ gives

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4\theta/2}{\sin^4\theta/2}+\frac{2}{\sin^2\theta/2\cos^2\theta/2}+\frac{1+\sin^4\theta/2}{\cos^4\theta/2}\right] \qquad (5.64)$$

where $\alpha=e^2/4\pi$. The first term in (5.64) comes from $\overline{|m_a|^2}$, the second term comes from Re $m_a m_b^*$, and the third term comes from $\overline{|m_b|^2}$. Note that

$$(p'_1-p_1)^2=-2p^2(1-\cos\theta)=-4p^2\sin^2\theta/2$$

$$(p'_2-p_1)^2=-2p^2(1+\cos\theta)=-4p^2\cos^2\theta/2$$

give rise to the denominators in (5.64).

For the electron-positron scattering, we also encounter two graphs (Figs. 5.4a,b), known as Bhabha amplitude. We can either work out the amplitude m explicitly, or make use of the crossing relation,

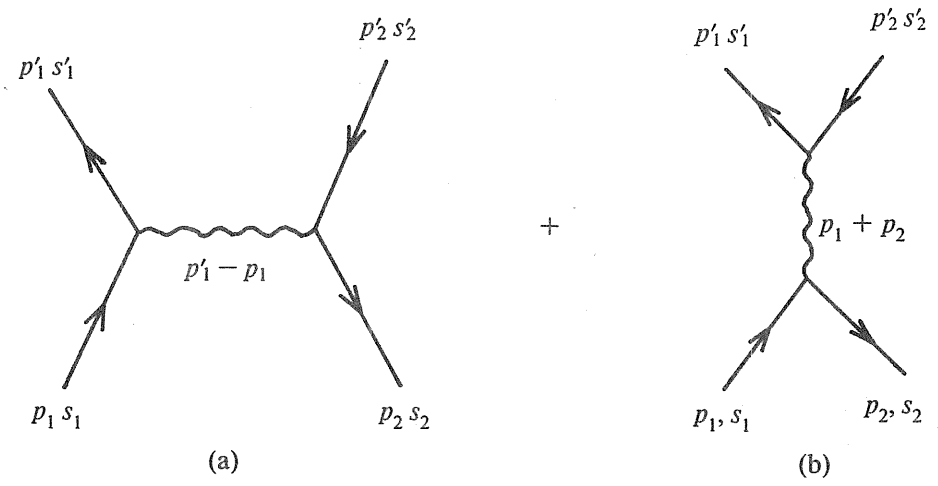


Fig. 5.4 Bhabha scattering.

$$|m_{\text{Fig. 5.4}}|^2 = |m_{\text{Fig. 5.3}}|^2 \, (p_2 \longleftrightarrow -p'_2).$$

However, we should realize that the c.m. frame of the e^-e^- scattering does not turn into the c.m. frame of the e^-e^+ system after the crossing. We have to compute the phase space factors of both processes explicitly. We shall not repeat the calculation here, and only quote the result. In the high energy limit, $E >> m$, we have

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4\theta/2}{\sin^4\theta/2} - \frac{2\cos^4\theta/2}{\sin^2\theta/2} + \frac{1+\cos^2\theta}{2}\right]. \tag{5.65}$$

The first term of (5.65) is the same as the first term of the e^-e^- scattering. The last term is due to the annihilation process alone. The middle term is due to the interference of Figs. 5.4a and 5.4b.

6. Interactions with the Classical Sources

There are two cases that we wish to study: The electromagnetic fields interacting with an external current j_μ, and the electrons interacting with a classical electromagnetic field.

(a) Electromagnetic fields interacting with an external current j_μ

In this case, we encounter the Lagrange function

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - A_\mu j_\mu \tag{5.66}$$

and the field equation

$$\partial_\mu F_{\mu\nu} = \partial^2 A_\nu - \partial_\mu \partial_\nu A_\mu = j_\nu \,. \tag{5.67}$$

In the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, we can obtain A_0 from j_0 via the Coulomb law. The total Hamiltonian of the system becomes

$$H = H_0 + \frac{1}{2} \int d^3x \, d^3y \, j_0 (x, t) \frac{1}{4\pi |\mathbf{x}-\mathbf{y}|} j_0 (y, t)$$

$$- \int d^3x \, \mathbf{A} \cdot \mathbf{j}. \tag{5.68}$$

In (5.68), H_0 is the free Coulomb gauge Hamiltonian. The second term in (5.68) has no dynamical variables and does not lead to any emission or absorption of photons. (In perturbation expansion, the contribution due to this term is cancelled term by term by the contribution due to the $\eta_\mu \eta_\nu$ term in the Coulomb gauge photon propagator. See discussion in Sec. 3.) Hence, we can ignore it. The last term of (5.68) leads to the emission of photons that we wish to study.

The S-matrix due to the last term

$$H_I = -\int d^3x \, \mathbf{A} \cdot \mathbf{J} \tag{5.69}$$

is

$$U(\infty, -\infty) = T \exp\left[-i \int dt \, H_I (t)\right]. \tag{5.70}$$

We can express the interaction representation variable $\mathbf{A}$ in terms of the creation and the annihilation operators

$$\mathbf{A}(x, t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 \, 2|k|}} \sum_\lambda \varepsilon(k, \lambda) \left(a(k, \lambda) \, e^{-ik\cdot x} + a^\dagger (k, \lambda) \, e^{ik\cdot x}\right), \tag{5.71}$$

obtaining

$$H_I = -\int d^3x \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2|k|}} \sum_\lambda \boldsymbol{\varepsilon}(k,\lambda)\,(a(k,\lambda)\, e^{-ik\cdot x} + a^\dagger(k,\lambda)\, e^{ik\cdot x}) \cdot \mathbf{j}(x,t) \tag{5.72}$$

$$= -\int \frac{d^3k}{\sqrt{(2\pi)^3\, 2|k|}} \sum_\lambda (j(k,\lambda,t)\, a^\dagger(k,\lambda) + j^*(k,\lambda,t)\, a(k,\lambda))$$

where

$$j(k,\lambda,t) \equiv \int d^3x\, e^{ik\cdot x}\, \boldsymbol{\varepsilon}(k,\lambda)\cdot \mathbf{j}(x,t). \tag{5.73}$$

Note that H_I is linear in the creation and the annihilation operators. Under this condition, it is possible to show that the T-order product (5.70) and the normal-order product introduced below are equal to each other to within a normalization factor,

$$S \equiv U(\infty, -\infty) = \mathrm{T}\exp\left\{ i \int dt \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\,|k|}} \times \sum_\lambda \left(j(k\lambda,t)\, a^\dagger(k\lambda) + j^*(k\lambda,t)\, a(k,\lambda)\right)\right\}$$

$$= N : \exp\left\{ i \int dt \int \frac{d^3k}{\sqrt{(2\pi)^3\, 2\,|k|}} \times \sum_\lambda \left(j(k\lambda,t)\, a^\dagger(k,\lambda) + j^*(k\lambda,t)\, a(k,\lambda)\right)\right\} :$$

$$= N : \exp\left\{ i \int d^3k \sum_\lambda \left(f(k,\lambda)\, a^\dagger(k,\lambda) + f^*(k,\lambda)\, a(k,\lambda)\right)\right\} :$$

$$= N \exp\left[i \int d^3k \sum_\lambda f(k,\lambda)\, a^\dagger(k,\lambda) \right] \exp\left[i \int d^3k \sum_\lambda f^k(k\lambda)\, a(k,\lambda)\right] \tag{5.74}$$

where

$$f(k,\lambda) \equiv \int dt \frac{j(k,\lambda,t)}{\sqrt{(2\pi)^3 2|k|}}$$

$$= \int d^4x \frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3 2|k|}} \varepsilon(k,\lambda)\cdot \mathbf{j}(x,t). \tag{5.75}$$

We may determine the magnitude of the normalization factor N by requiring $U(\infty, -\infty)$ to be unitary, giving

$$|N| = \exp\left[-\frac{1}{2}\int d^3k |f(k,\lambda)|^2 \right]. \tag{5.76}$$

If we begin with the vacuum state $|0\rangle$ obeying $a(k,\lambda)|0\rangle = 0$; then, we end up with the final state $|\ \rangle^j$ in the presence of the classical current j, giving

$$|\rangle^j \equiv S|0\rangle$$

$$= N \exp\left[i \int d^3k \sum_\lambda f(k,\lambda)\, a^\dagger(k,\lambda) \right] |0\rangle. \tag{5.77}$$

This new state $|\rangle^j$ is an eigenstate of the annihilation operator $a(k,\lambda)$,

$$a(k,\lambda)|\rangle^j = a(k,\lambda) S|0\rangle$$

$$= S[S^{-1} a(k,\lambda) S]|0\rangle$$

$$= S(a(k,\lambda) + if(k,\lambda))|0\rangle$$

$$= if(k,\lambda)|\rangle^j \tag{5.78}$$

with the eigenvalue $if(k,\lambda)$. This state is known as a coherent state. This coherent state contains an indefinite number of photons. One can verify that the number of photons emitted by a classical current obeys the Poisson distribution.

(b) Electrons interact with a classical electromagnetic field

In this case, we assume that the Maxwell field is known. The Lagrange function of the system is

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - e A\!\!\!/^{\text{ext}} - m)\psi \tag{5.79}$$

where A^{ext} is the known classical Maxwell potential. The field equations for ψ and $\bar{\psi}$ are

$$(i\partial\!\!\!/ - e A\!\!\!/^{\text{ext}} - m)\,\psi = 0 \tag{5.80a}$$

$$\bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - e A\!\!\!/^{\text{ext}} - m) = 0\,. \tag{5.80b}$$

In principle, we can solve Eq. (5.80), and expand ψ and $\bar{\psi}$ in terms of the 1-particle solutions of (5.80). If the coupling constant is small (in QED, $\alpha = e^2/4\pi = 1/137$ is indeed small), we can use the Dyson-Wick expansion to develop a set of Feynman rules. Consider

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\text{int}} \tag{5.81}$$

$$\mathscr{L}_0 = \bar{\psi}(i\partial\!\!\!/ - m)\,\psi \tag{5.82}$$

$$\mathscr{L}_{\text{int}} = -e\,\bar{\psi} A\!\!\!/^{\text{ext}}\,\psi = -\mathscr{H}_{\text{int}}. \tag{5.83}$$

In the present case, the electron-external-field vertex is given by Fig. 5.5,

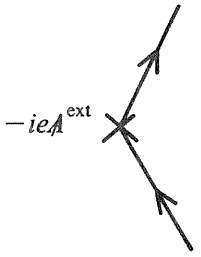


Fig. 5.5 The electron-external-field vertex.

where $A_\mu^{\text{ext}}(x)$ is a given function of x. We still have the same electron propagator as in QED, but no longer have the photon propagator. As an example, let us work out the Coulomb scattering problem. Here, we have an external field,

$$A_0(x) = \frac{-Ze}{4\pi|x|}, \qquad \mathbf{A}(x) = 0. \tag{5.84}$$

In the momentum space, the fourier transform of A_0 is

$$\int d^4x\, e^{-iq\cdot x} A_0(x, t) = -\frac{Ze\,(2\pi)\delta(q_0)}{|\mathbf{q}^2|}. \tag{5.85}$$

Thus, the invariant amplitude for the Coulomb scattering process in Fig. 5.6 is

$$(-ie)\,\gamma_0 \frac{2\pi\delta(E_f - E_i)(-Ze)}{|\mathbf{q}^2|}$$

Fig. 5.6 Coulomb scattering.

$$\mathcal{M} = -\frac{Ze\,\bar{u}(p_f, s_f)\,(-ie)\,\gamma_0\,u(p_i, s_i)}{|\mathbf{q}^2|}. \tag{5.86}$$

The S-matrix of this process is

$$S_{fi} = \sqrt{\frac{m}{VE_i}}\sqrt{\frac{m}{VE_f}}\,\mathcal{M}(2\pi)\delta(E_f - E_i). \tag{5.87}$$

Note that we do not have the momentum conservation any more because the x-dependent external potential $A_0^{\text{ext}}(x)$ destroys the translational invariance of the system.

We can work out the differential cross section in a similar way. It is

$$d\sigma = \frac{m}{E_i v_i}|\mathcal{M}|^2 \frac{m}{E_f}\frac{d^3P_f}{(2\pi)^3}\,2\pi\delta\,(E_f - E_i). \tag{5.88}$$

Indeed, we can recover the cross section (5.88) from an electron-nucleus scattering provided that we can treat the nucleus (particle b) as an infinitely heavy Dirac particle ($M \to \infty$). Then we have

$$\frac{m}{E_a}\frac{M}{E_b} \to \frac{m}{E_a},$$

$$\frac{m}{E'_a}\frac{M}{E'_b}\frac{d^3p'_a}{(2\pi)^3}\frac{d^3p'_b}{(2\pi)^3}(2\pi)^4\,\delta^4\,(p'_a + p'_b - p_a - p_b) \to \frac{m}{E'_a}\frac{d^3p'_a}{(2\pi)^3}\,2\pi\delta\,(E'_a - E_a),$$

$$v_{ab} \to v_a,$$

which are precisely the kinematical and the phase space factors in (5.88).

Carrying out the traces and other kinematics, we obtain the Mott cross section formula in the lab frame as

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{4p^2\beta^2 \sin^4 \theta/2}(1 - \beta^2 \sin^2 \theta/2) \tag{5.89}$$

with

$$\beta = P_i/E_i.$$

7. Compton Scattering and the Low Energy Theorem

The Compton scattering describes the elastic scattering of a photon on a free electron. The lowest order diagrams are shown in Fig. 5.7.

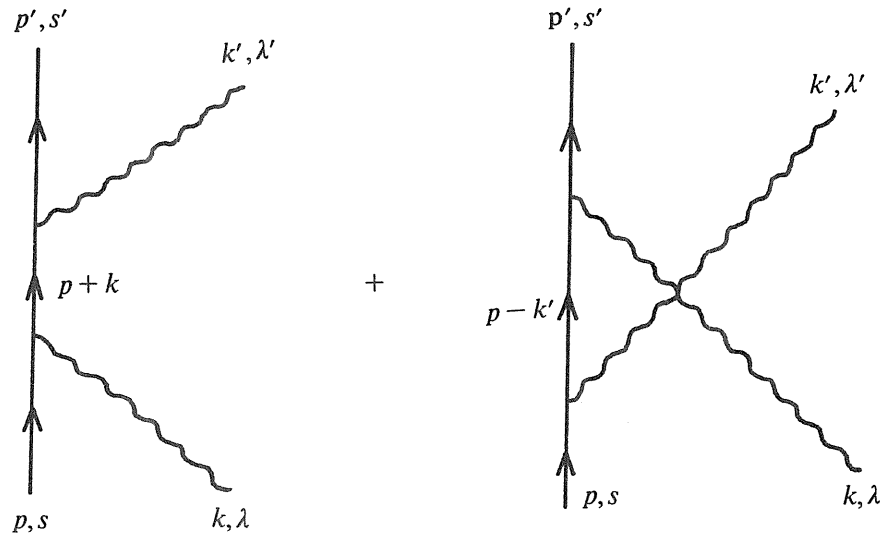


Fig. 5.7 Compton scattering.

We do not have time to work out the details here. However, we wish to point out a few important features of the process.

The invariant amplitudes for the process in Figs. 5.7a, b are

$$\mathcal{M}_a = \bar{u}(p', s')(-ie\not{\varepsilon}')\frac{i}{\not{p}+\not{k}-m+i\varepsilon}(-ie\not{\varepsilon})\,u(p, s) \tag{5.90a}$$

$$\mathcal{M}_b = \bar{u}(p', s')(-ie\not{\varepsilon})\frac{i}{\not{p}-\not{k}'-m+i\varepsilon}(-ie\not{\varepsilon}')\,u(p, s) \tag{5.90b}$$

and the sum is

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b. \tag{5.90c}$$

It is easy to see that, as k and $k' \to 0$, the electron propagators in Fig. 5.7 are of $O(1/k)$. These amplitudes are called the pole terms and are dominant at the low energy. For the unpolarized initial electrons and for detectors which do not detect the final electron spins, the cross section in the lab frame is given by the Klein-Nishina formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2}\left(\frac{k'}{k}\right)^2\left[\frac{k'}{k} + \frac{k}{k'} + 4\,(\varepsilon\cdot\varepsilon')^2 - 2\right]. \tag{5.91}$$

In the $k \to 0$ limit, we find $k'/k = 1$ and

$$\left(\frac{d\sigma}{d\Omega}\right)_{k\to 0} = \frac{\alpha^2}{m^2}(\varepsilon\cdot\varepsilon')^2. \tag{5.92}$$

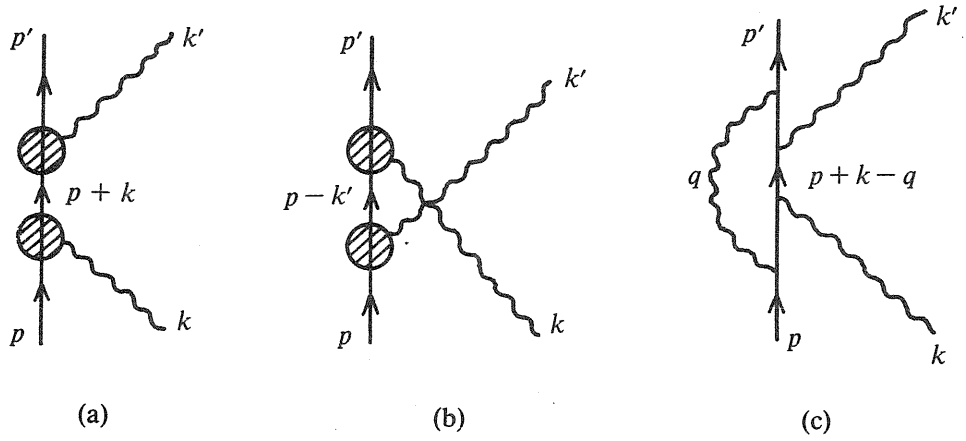


Fig. 5.8 Higher order contributions to the Compton scattering: (a) and (b) pole terms, and (c) not a pole term.

In Fig. 5.8, we look at the higher order contributions to the Compton scattering. The electron pole propagators appear in diagrams 5.8a and b, but not in diagrams such as Fig. 5.8c. At k and $k' \to 0$, the pole terms dominate.Note that Fig. 5.8a and b are the same as Fig. 5.7a and b with the inclusion of form factor contributions.

The virtual electrons in the pole terms are nearly on the mass-shell. For diagram 5.8a (or 5.7a), we have

$$(p+k)^2 - m^2 = p^2 + 2p\cdot k + k^2 - m^2 = 2p\cdot k \tag{5.93}$$

where we have used the fact that $p^2 = m^2$ and $k^2 = 0$. In other words, the violation of the mass-shell condition is $O(k)$. Another way of looking at

the mass-shell condition is to compute the difference between the actual energy p_0+k_0, and its on-shell energy,

$$E(p+k) \equiv \sqrt{(\mathbf{p}+\mathbf{k})^2 + m^2}, \tag{5.94}$$

giving

$$p_0+k_0 - E(p+k) = \frac{(p+k)^2 - m^2}{p_0+k_0 + E(p+k)}$$

$$= \frac{2p\cdot k}{p_0+k_0 + E(p+k)} = O(k). \tag{5.95}$$

The "energy violation" is $O(k)$, and according to the uncertainty principle, the virtual state can live a time $O(1/k)$. The long-lived virtual states are the pole terms with the form factor contributions included.

Equation (5.92) is the classical result for the Thomson scattering. We can actually show that (5.92) is also true for a photon-proton or a photon-$\pi^{\pm}$ scattering even though the proton and the pions are also subject to strong interactions. As $k \to 0$, the photon has a large wave length. It can only detect the total charge. The charge of a proton or a $\pi^{\pm}$ is the same as that of an e^-. Hence, we have the same low energy limits as given by (5.90) and (5.92). In fact, if we work out the next order correction, we find that the Compton scattering amplitude is determined completely by the charge (to $O(1/k)$) and the magnetic moment (to $O(1)$) of the scatterers, consistent with the physical picture shown in Figs. 5.8a, b. This result is known as the low energy theorem of the Compton scattering.

Chapter 6

HIGHER ORDER PROCESSES

1. Degrees of Divergence

To study the higher order processes, we introduce the concept of the superficial degrees of divergence D for a diagram. In computing the degrees of divergence, we just count any loop integral as p^4, an electron propagator as $1/p$, and a photon propagator as $1/p^2$. The algebraic sum of these powers gives the degree of divergence D. We call a diagram with $D = 2$ quadratically divergent, a diagram with $D = 1$ linearly divergent, and a diagram with $D = 0$ logarithmically divergent. Diagrams with $D < 0$ are convergent. For a theory such as QED where the coupling constants are dimensionless, the degrees of divergence depend only on the numbers of the external legs (particles), independent of the internal structure of the diagrams. For QED, we can express D as

$$D = 4 - \frac{3}{2}F_e - B_e \tag{6.1}$$

where

F_e = number of external electron lines
B_e = number of external photon lines.

Since F_e and B_e are nonnegative integers, we only have a finite number of classes of divergent diagrams.

Using D as a test of divergence is not always reliable. Sometimes we may encounter a diagram with $D < 0$ but having a divergent subdiagram. On the other hand, we also encounter diagrams with $D > 0$ but with divergent parts cancelled. However, we know how to handle these exceptional cases.

In Table II, we list all possible sets of superficial divergent diagrams in QED. We have ruled out two of them by the Furry's theorem. In the following, we shall describe briefly the meaning of the Furry's theorem.

Table II. Superficially divergent diagrams in QED.

B_e	F_e	D	Remark
0	0	4	Vacuum diagrams
1	0	3	Vanishes by Furry Theorem
2	0	2	Vacuum polarization diagram
3	0	1	Vanishes by Furry Theorem
4	0	0	Actually finite
0	2	1	Electron self-energy diagrams
1	2	0	Vertex correction diagrams

Furry's Theorem: Diagrams containing an electron loop with an odd number of photons attached should vanish.

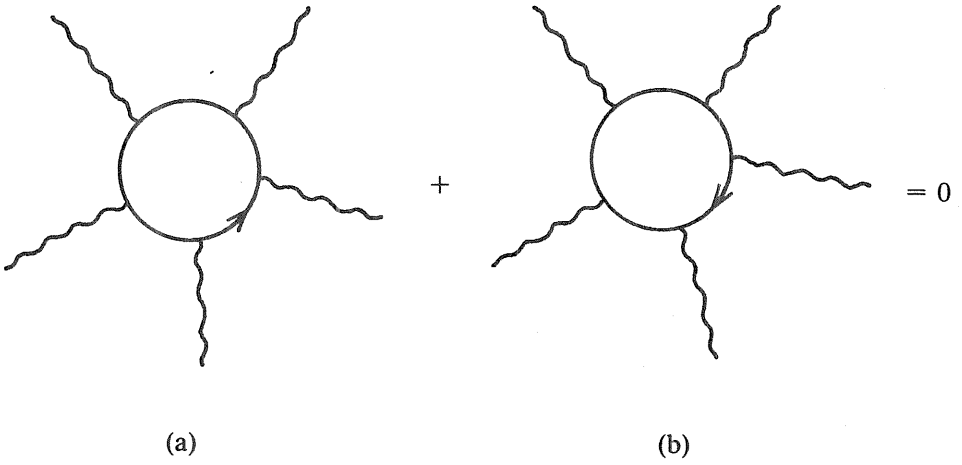


Fig.6.1 According to Furry's theorem, the contributions due to diagrams (a) and (b) cancel.

This theorem follows from the fact that the electron contribution to the loop and the positron contribution to the loop (the same diagram with the arrows pointing in the opposite direction) cancel each other.

In QED, we only encounter four classes of divergent diagrams: the vacuum diagrams, the vacuum polarization diagrams, the electron self-energy diagrams, and the vertex correction diagrams. We have already known the meaning of the vacuum diagrams. We shall consider the other divergent diagrams briefly.

2. The Vacuum Polarization Diagrams

Figure 6.2 represents the simplest vacuum polarization diagram. Its amplitude is

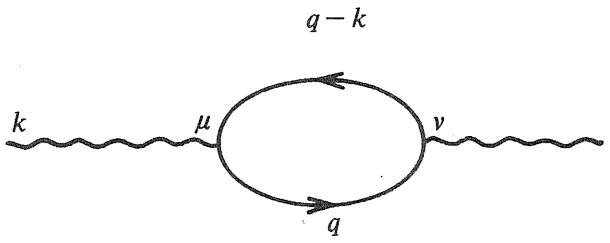


Fig. 6.2 The lowest order vacuum polarization diagram.

$$\Pi_{\mu\nu} = \int \frac{d^4q}{(2\pi)^4}(-1)\,\mathrm{Tr}\left[(-ie\gamma_\mu)\frac{i}{\not q - \not k - m + i\varepsilon}\right. \\ \left.\times(-ie\gamma_\nu)\frac{i}{\not q - m + i\varepsilon}\right]. \tag{6.2}$$

A simple power counting of $\Pi_{\mu\nu}$ leads to a quadratic divergence. The presence of the ultraviolet divergences indicates that our understanding of QED at the short distance is incomplete. In a complete theory, these divergences may not appear.

Luckily, in QED as in a number of other theories, we are able to describe all the physical processes in terms of a set of measurable quantities such as the charge and the mass of a physical electron. This method of bypassing the difficulties of ultraviolet divergences is known as the renormalization technique.

We first consider a regulated vacuum polarization amplitude, $\Pi^R_{\mu\nu}$, which depends on some ultraviolet cutoff mass scale Λ.The details about how to implement a cutoff shceme is not important. But we require that the regulation scheme is consistent with both the Lorentz invariance and the gauge invariance.

The gauge invariance implies that

$$k_\mu \Pi^R_{\mu\nu}(k) = 0. \tag{6.3}$$

Equation (6.3), together with the Lorentz invariance, leads to

$$\Pi^R_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu}k^2)\,\Pi^R(k^2). \tag{6.4}$$

A simple power counting indicates that Π^R only diverges logarithmically.

Pauli and Villars introduced a regulation scheme as follows: To obtain

$\Pi^R_{\mu\nu}$, we subtract from (6.2) expressions corresponding to the "heavy electrons", giving

$$\Pi^R_{\mu\nu}(k) \equiv \Pi_{\mu\nu}(k, m) - \sum_i C_i \Pi_{\mu\nu}(k, M_i) \tag{6.5}$$

where C_i's are chosen to make the k-integration converge. The M_i's play the role of the cutoff Λ.

We can provide the following interpretation for the Pauli-Villars scheme. The subtraction is equivalent to replacing the original Lagrange function by a regulated one via

$$\mathscr{L}_{\text{reg}} = \mathscr{L}(\varepsilon.m.) + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi$$

$$- \sum_i C_i \bar{\psi}_i (i\partial\!\!\!/ - eA\!\!\!/ - M_i)\psi_i$$

where ψ_i's correspond to a set of fictitious heavy electron field variables with masses M_i's. Both the gauge invariance and the Lorentz invariance are preserved in $\mathscr{L}_{\text{reg}}$.

To make $\Pi^R_{\mu\nu}$ converge, we require

$$\Sigma C_i = 1 \tag{6.6a}$$

$$\Sigma C_i M_i^2 = m^2 . \tag{6.6b}$$

In order to satisfy (6.6), we need a minimum of two M's. Then, we have

$$\Pi^R_{\mu\nu} = -e^2 \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{\text{Tr}\,[\gamma_\mu (q\!\!\!/ - k\!\!\!/ + m)\gamma_\nu (q\!\!\!/ + m)]}{((q-k)^2 - m^2 + i\varepsilon)(q^2 - m^2 + i\varepsilon)} \right.$$

$$\left. - \text{regulator terms} \right\} . \tag{6.7}$$

There are a number of standard tricks for handling this kind of integrals.

Trick #1: We introduce the Feynman parameter representation,

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} \tag{6.8a}$$

and, more generally,

$$\frac{1}{A_1 A_2 \ldots A_n} = (n-1)! \int_0^1 \frac{d\alpha_1 d\alpha_2 \ldots d\alpha_n \, \delta(\alpha_1+\alpha_2+\ldots+\alpha_n-1)}{(\alpha_1 A_1+\alpha_2 A_2+\ldots+\alpha_n A_n)^n} \tag{6.8b}$$

With the help of (6.8a), we can express (6.7) as

$$\Pi^R_{\mu\nu} = -4e^2 \int \frac{d^4q}{(2\pi)^4} \int_0^1 dx \left\{ \frac{q_\mu (q-k)_\nu + q_\nu (q-k)_\mu - g_{\mu\nu} q\cdot(q-k) + m^2 g_{\mu\nu}}{[x((q-k)^2-m^2)+(1-x)(q^2-m^2)+i\varepsilon]^2} \right.$$

$$\left. - \text{ regulator terms} \right\} . \tag{6.9}$$

$$= -4e^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{q_\mu (q-k)_\nu + q_\nu (q-k)_\mu - g_{\mu\nu} q\cdot(q-k) + m^2 g_{\mu\nu}}{[(q-xk)^2+x(1-x)k^2-m^2+i\varepsilon]^2} \right.$$

$$\left. - \text{ regulator terms} \right\} .$$

Since the regulated $\Pi^R_{\mu\nu}$ is convergent, we can make a translation of the origin and introduce the new variable,

$$q' = q - xk .$$

In term of q', the denominator is a function of $(q')^2$ only. Then, we can ignore terms in the numerator which are linear in q'. We can also replace $q'_\mu q'_\nu$ terms in the numerator by $\frac{1}{4} g_{\mu\nu}(q')^2$. After all these simplifications, we have

$$\Pi^R_{\mu\nu}(k) = -4e^2 \int_0^1 dx \int \frac{d^4q'}{(2\pi)^4}$$

$$\times \left\{ \frac{-\frac{1}{2} g_{\mu\nu} q'^2 - 2x(1-x)\, k_\mu k_\nu + g_{\mu\nu}\, x(1-x)k^2 + m^2 g_{\mu\nu}}{[q'^2+x(1-x)k^2-m^2+i\varepsilon]^2} \right.$$

$$\left. - \text{ regulator terms} \right\} . \tag{6.10}$$

For notational simplicity, we shall omit the primes on q.
Equation (6.10) looks very similar to a spherically symmetric integration. In fact, we can bring (6.10) into an Euclidean form by the second trick described below.

Trick #2: We can make a 90° rotation in the complex q_0-plane (called the Wick rotation), and achieve an Euclidean integration.

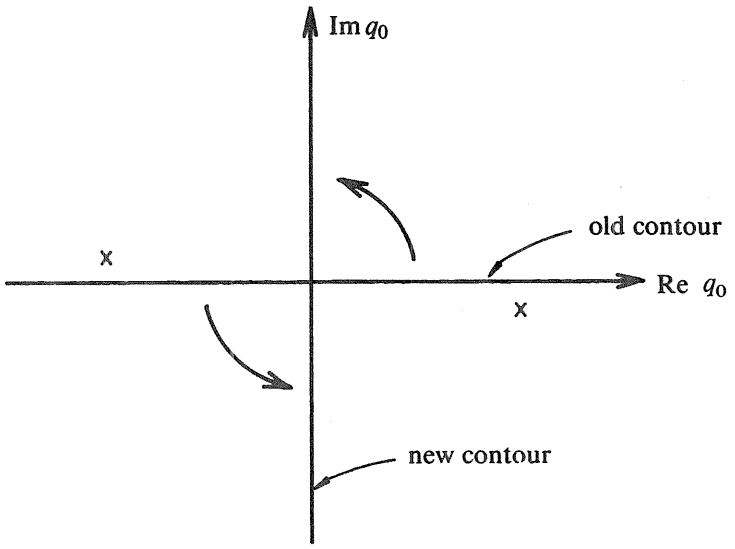


Fig. 6.3 The Wick rotation in the complex q-plane.

Since there are no singularities in the first and the third quadrants, we can make the rotation as shown in Fig. 6.3. After the rotation, we have

$$\begin{aligned} q_0 &= iq_4 \\ q^2 &= -(q^2) \text{ Euclidean} \\ d^4q &= i(d^4q) \text{ Euclidean,} \end{aligned}$$

and consequently,

$$\Pi^R_{\mu\nu}(k) = -4ie^2 \int_0^1 dx \int \frac{d^4q}{(2\pi)^4}$$

$$\times \left\{ \frac{\frac{1}{2}g_{\mu\nu}q^2 - 2x(1-x)\,k_\mu k_\nu + g_{\mu\nu}\,x(1-x)k^2 + m^2 g_{\mu\nu}}{(-q^2+x(1-x)k^2-m^2)^2} \right.$$

$$\left. - \text{ regulator terms} \right\}$$

$$= \frac{-i\alpha}{\pi}\int_0^1 dx \int_0^\infty 2q^3 dq \left\{ \frac{\frac{1}{2}g_{\mu\nu}q^2 - 2x(1-x)\,k_\mu k_\nu + g_{\mu\nu}\,x(1-x)k^2 + m^2 g_{\mu\nu}}{(q^2+m^2-x(1-x)k^2)^2} \right.$$

$$\left. - \text{ regulator terms} \right\} \tag{6.11}$$

with

$$\alpha = e^2/4\pi.$$

The Euclidean integrations are elementary,

$$\int_0^\Lambda \frac{q^4\, d(q^2)}{(q^2+c^2)^2} = \Lambda^2 - 2c^2 \ln\frac{\Lambda^2}{c^2} + c^2$$

$$\int_0^\Lambda \frac{q^2\, d(q^2)}{(q^2+c^2)^2} = \ln\frac{\Lambda^2}{c^2} - 1.$$

Under the constraint (6.6), the Λ^2 and the constant terms in the above integrals cancel. We only need to consider the logarithmic term, and obtain,

$$\Pi^R_{\mu\nu} = -\frac{i\alpha}{\pi}\int_0^1 dx \left\{ g_{\mu\nu}(-m^2+x(1-x)k^2)\ln\frac{\Lambda^2}{m^2-x(1-x)k^2} \right.$$

$$+ (-2x(1-x)k_\mu k_\nu + g_{\mu\nu}\,x(1-x)\,k^2 + m^2 g_{\mu\nu})\ln\frac{\Lambda^2}{m^2-x(1-x)k^2}$$

$$\left. - C_1(m\to M_1) - C_2(m\to M_2) \right\}$$

$$= (k_\mu k_2 - g_{\mu\nu}k^2)\frac{2i\alpha}{\pi}\int_0^1 x(1-x)dx$$

$$\times\left\{ \ln\frac{\Lambda^2}{m^2-x(1-x)k^2} - C_1\ln\frac{\Lambda^2}{M_1^2-x(1-x)k^2} - C_2\ln\frac{\Lambda^2}{M_2^2-x(1-x)k^2} \right\}$$

$$= (k_\mu k_\nu - g_{\mu\nu}k^2)\,\Pi^R(k^2) \tag{6.12}$$

with, at $M_i^2 >> m^2$,

$$\Pi^R(k^2) = \frac{2i\alpha}{\pi}\int_0^1 x(1-x)\,dx\left[\ln\frac{m^2}{m^2-x(1-x)k^2}\right.$$

$$\left. - C_1\ln\frac{m^2}{M_1^2-x(1-x)k^2} - C_2\ln\frac{m^2}{M_2^2-x(1-x)k^2}\right]$$

$$= \frac{2i\alpha}{\pi}\int_0^1 x(1-x)\,dx\left[\ln\frac{m^2}{m^2-x(1-x)k^2}\right.$$

$$\left. + (C_1\ln M_2^2/m^2 + C_2\ln M_2^2/m^2) + O(1/M^2)\right]$$

$$= \frac{i\alpha}{3\pi}\left[\ln M^2/m^2 - 6\int_0^1 dx\,x(1-x)\ln\left(1-x(1-x)k^2/m^2\right)\right]. \tag{6.13}$$

It is pleasing to see that the factor $k_\mu k_\nu - g_{\mu\nu}k^2$ emerges automatically. The only dependence on the cutoff masses is through the combination,

$$\ln M^2/m^2 \equiv \Sigma\, C_i \ln M_i^2/m^2.$$

Fig. 6.4 The modified photon propagator represented as a sum of vacuum polarizations.

The vacuum polarization in Fig. 6.4 modifies the photon propagator to

$$iD(k)_{\mu\nu} = \frac{-ig_{\mu\nu}}{k^2} + \frac{-i}{k^2}\Pi^R_{\mu\nu}\frac{-i}{k^2} + \frac{-i}{k^2}\Pi^R_{\mu\lambda}\frac{-i}{k^2}\Pi^R_{\lambda\nu}\frac{-i}{k^2} + \ldots$$

$$= \frac{-ig_{\mu\nu}}{k^2}\left[1 + i\Pi^R(k^2) + (i\Pi^R)^2 + \ldots\right] + k_\mu k_\nu \text{ terms}$$

$$= \frac{-ig_{\mu\nu}}{k^2}\left[1-\frac{\alpha}{3\pi}\ln\frac{M^2}{m^2}+\frac{2\alpha}{\pi}\int dx\, x(1-x)\ln\left(1-x(1-x)k^2/m^2\right)\right.$$

$$\left.+\ \ldots\right]+k_\mu k_\nu \text{ terms}$$

$$= \frac{-ig_{\mu\nu}}{k^2}Z_3\left[1+\frac{2\alpha}{\pi}\int dx\, x(1-x)\ln\left(1-x(1-x)k^2/m^2\right)\right.$$

$$\left.+\ O(\alpha^2)\right]+k_\mu k_\nu \text{ terms}$$

where

$$Z_3 = 1-\frac{\alpha}{2\pi}\ln M^2/m^2 + O(\alpha^2)$$

is called the photon wave function renormalization constant. Here, we make a distinction between the bare charge (coupling) e_0 and the physical charge e. The charge that we observe is not e_0, but

$$e = e_0\sqrt{Z_3}. \tag{6.14}$$

(In a more complete description, we should write (6.14) as

$$e = e_0 Z_1^{-1} Z_2 \sqrt{Z_3}.$$

Thanks to the Ward identity to be discussed below, we have $Z_1 = Z_2$ and Eq. (6.14) prevails. See Sec. 4 below.) Since e is a measured quantity ($e^2/4\pi = 1/137$) and Z_3 depends on the cutoff M, we conclude that the bare charge e_0 is cutoff-dependent,

$$e_0(M) = e/\sqrt{Z_3} = e(1+\frac{\alpha}{6\pi}\ln M^2/m^2 + O(\alpha^2))$$

or

$$\alpha_0(M) = \alpha\,(1+\frac{\alpha}{3\pi}\ln M^2/m^2 + O(\alpha^2)). \tag{6.15}$$

We can interpret $\alpha_0(M)$ as an effective (running) coupling constant at the momentum scale M or, equivalently, at the distance scale $1/M$.

To see how large the logarithmic correction is, we compute the mass M such that α_0 increases by 10%,

$$\frac{\alpha}{3\pi} \ln M^2/m^2 = 0.1$$

with

$$\alpha = 1/137.036.$$

Hence, we have

$$\frac{M^2}{m^2} = \exp(0.1 \times 3\pi \times 137.036) = 1.23 \times 10^{56}$$

or

$$M = m \times 1.11 \times 10^{28} = 5.67 \times 10^{27}\,\text{MeV} = 10g.$$

This mass is so large that it is only of academic interest. The gravitation force due to M's is 3×10^{13} stronger than the Coulomb force at the same distance.

We may interpret $e < e_0$ as a screening effect. The vacuum polarization tends to shield the charge as we go away from a point charge.

The vacuum polarization also modifies the static potential between two point charges. Its fourier transform becomes

$$U(k) \equiv \frac{e^2}{\mathbf{k}^2}\left[1 + \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x) \ln\left(1+x(1-x)k^2/m^2\right) + \ldots\right] \tag{6.16}$$

which, at small k, reduces to

$$\frac{e^2}{\mathbf{k}^2}\left[1 + \frac{\alpha}{15\pi}\frac{\mathbf{k}^2}{m^2} + O(\alpha^2)\right].$$

In the position space, it gives rise to a potential

$$V(r) \equiv \left(1 - \frac{\alpha}{15\pi}\frac{\nabla^2}{m^2}\right)\frac{e^2}{4\pi r} = \frac{e^2}{4\pi r} + \frac{\alpha e^2}{15\pi m^2}\delta^3(x). \qquad (6.17)$$

This additional term will change the s-state energy relative to the p-state energy. This effect, among other contributions, is indeed observed in the Lamb shift.

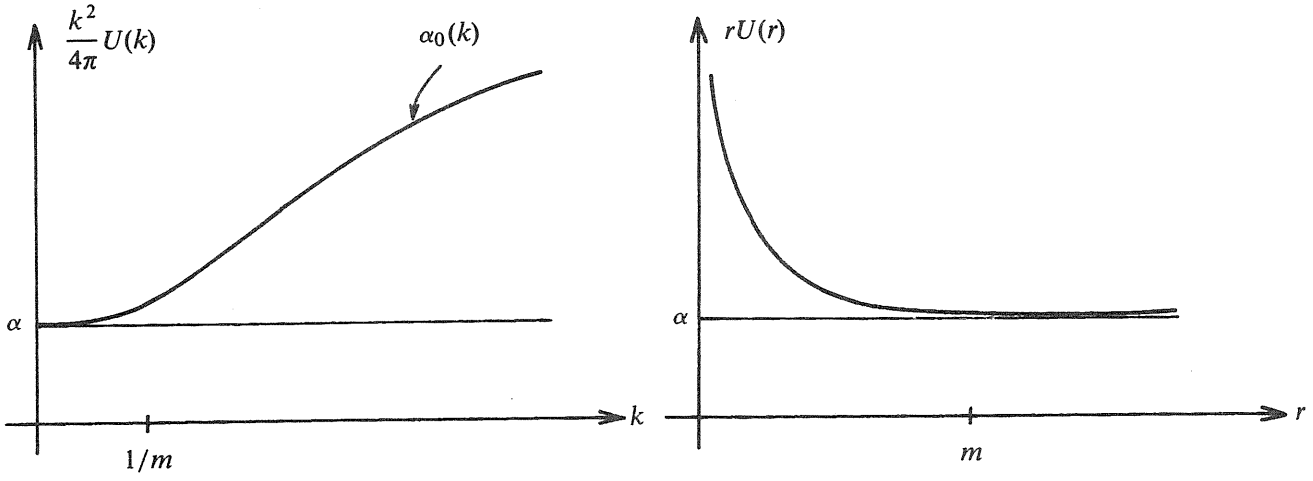


Fig. 6.5 (a) The modified potential $k^2U(k)$ as a function of k, and (b) $rU(r)$ as a function of r.

In Fig. 6.5, we sketch the potential described in (6.16). Figs. 6.5a, b represent the deviation of Coulomb law, and give rise to a k and x dependent effective charge.

We wish to point out that a repeated sum of the vacuum polarization diagrams in Fig. 6.4 leads to a geometric series

$$\frac{-ig_{\mu\nu}}{k^2}[1 + i\Pi^R + (i\Pi^R)^2 + \ldots] + k_\mu k_\mu \text{ terms}$$

$$= \frac{-ig_{\mu\nu}}{k^2(1 - i\Pi^R)} + k_\mu k_\mu \text{ terms}.$$

This implies that the running coupling constant $\alpha(M)$ to this order obeys

$$\frac{1}{\alpha(M)} = \frac{1}{\alpha}\left(1 - \frac{\alpha}{3\pi}\ln\frac{M^2}{m^2}\right) = \frac{1}{\alpha} - \frac{1}{3\pi}\ln\frac{M^2}{m^2}.$$

It is interesting to see that the $1/\alpha(M)$ for two different cutoff masses

M_1 and M_2 are related by a similar relation

$$\frac{1}{\alpha(M_1)} = \frac{1}{\alpha(M_2)} - \frac{1}{3\pi} \ln \frac{M_1^2}{M_2^2}.$$

This is the type of relation derivable from the renormalization group method.

3. Electron Self-energy Diagrams

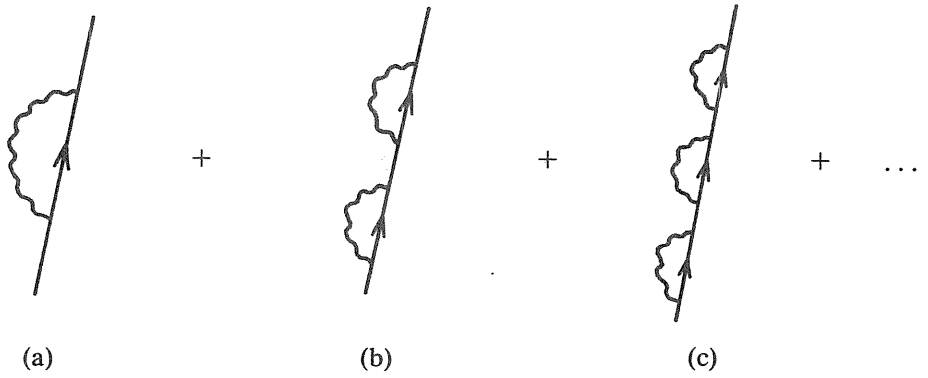


Fig. 6.6 Sum of electron self-energy diagrams.

An electron self-energy diagram consists of two external fermion lines but with no external photon lines. We call a self-energy diagram a proper self-energy diagram if it cannot be separated into two self-energy diagrams by cutting one internal fermion line. Otherwise, it is an improper self-energy diagram. In Fig. 6.6, diagram (a) is a proper self-energy diagram, and diagrams (b) and (c) are improper self-energy diagrams.

If we denote a proper self-energy diagram by $-i\Sigma(p)$, then, the sum of itself and all improper self-energy diagrams generated by it, as indicated in Fig. 6.6, gives

$$\begin{aligned} iS'_{\mathrm{F}}(p) &= \frac{i}{\not{p} - m_0} + \frac{i}{\not{p} - m_0}(-i\Sigma)\frac{i}{\not{p} - m_0} + \ldots \\ &= \frac{i}{\not{p} - m_0 - \Sigma(\not{p}) + i\varepsilon} \\ &= \frac{i}{\not{p} - m - (\Sigma(\not{p}) - \delta m) + i\varepsilon} \end{aligned} \tag{6.18}$$

where

$$\delta m \equiv m - m_0 \,. \tag{6.19}$$

In (6.18), m describes the physical mass, and m_0 describes the bare mass. The physical mass m determines how a particle propagates at large distances. The bare mass m_0 and the bare propagator $i/(\not{p}-m_0)$ describe how a particle propagates at short distances. Note that m is a measurable quantity, but m_0 is not. Just as e_0, m_0 also depends on the cutoff Λ.

In the following, we shall describe how to handle the self-energy correction without getting into the details.

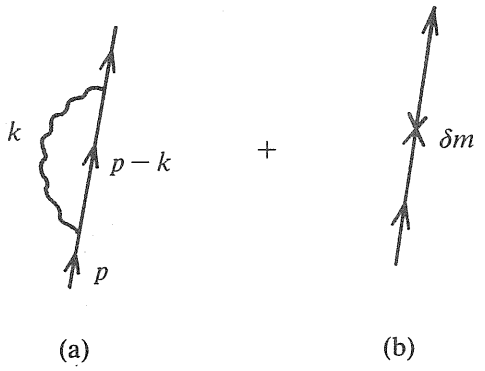


Fig. 6.7 (a) The lowest order electron proper self-energy diagram, and (b) a mass counter term.

The lowest order electron self-energy diagram is given in Fig. 6.7(a) and has the amplitude

$$-i\Sigma(\not{p}) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 - \lambda^2 + i\varepsilon} \gamma_\nu \frac{i}{\not{p}-\not{k}-m+i\varepsilon} \gamma_\nu \tag{6.20}$$

where λ is a fictitious photon mass that we introduce to control the infrared divergence. It is easy to see that $\Sigma(\not{p})$ is linearly divergent at $k \to \infty$. We choose the bare mass (or δm) such that $iS'_F(p)$ has a pole at $\not{p} = m$, giving

$$\Sigma(\not{p})\big|_{\not{p}=m} - \delta m = 0. \tag{6.21}$$

We have

$$\Sigma(\not p) - \delta m = \Sigma(\not p) - \Sigma(\not p)\big|_{\not p = m}$$

$$= -ie^2 \left\{ \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - \lambda^2 + i\varepsilon} \gamma_\nu \frac{1}{\not p - \not k - m + i\varepsilon} \gamma_\nu - (\text{''})\big|_{\not p = m} \right\}. \tag{6.22}$$

We find that $\Sigma(\not p) - \delta m$ is only logarithmically divergent. Since $\Sigma(\not p) - \delta m$ vanishes at $\not p = m$, we expect that we can factor out a $(\not p - m)$ factor. After some straightforward algebra, we obtain

$$-(\Sigma(\not p) - \delta m) = (\not p - m)\left[\frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} - \frac{\alpha}{2\pi} \ln \frac{m^2}{\lambda^2} + C(\not p) \right] \tag{6.23}$$

where Λ is an ultraviolet cutoff, and

$$C(\not p) = \frac{\alpha}{2\pi} \int_0^1 dx \left[\frac{2m^2 x(1-x)}{x^2 m^2 + \lambda^2 (1-x)} \right.$$

$$\left. - \frac{2m - (1-x)\not p}{\not p - m} \ln \frac{x^2 m^2 + \lambda^2 (1-x)}{x m^2 + (1-x)\lambda^2 - x(1-x)p^2} \right]. \tag{6.24}$$

We shall not worry about the detailed structure of $C(\not p)$. We only need to know that, by construction, $C(\not p)$ is finite and vanishes at $\not p = m$. Hence, we have

$$iS'_{\mathrm{F}}(p) = \frac{i}{(\not p - m)\left[1 + \frac{\alpha}{4\pi} \ln \frac{M^2}{m^2} - \frac{\alpha}{2\pi} \ln \frac{m^2}{\lambda^2} + C(\not p) \right]}$$

$$= \frac{i}{(\not p - m)\left(1 + \frac{\alpha}{4\pi} \ln \frac{M^2}{m^2} - \frac{\alpha}{2\pi} \ln \frac{m^2}{\lambda^2} \right)(1 + C(\not p) + O(\alpha^2))}$$

$$= \frac{iZ_2}{(\not p - m)(1 + C(\not p) + O(\alpha^2))} \tag{6.25}$$

with

$$Z_2^{-1} = 1 + \frac{\alpha}{2\pi}\left(\ln\frac{M^2}{m^2} - 2\ln\frac{m^2}{\lambda^2}\right). \tag{6.26}$$

Z_2 is known as the wave-function renormalization constant of an electron. It describes the probability of finding a bare electron alone inside a physical electron. Since the definition of a bare electron depends on the short and the long-distance resolutions ($1/\Lambda$ and $1/\lambda$), both Z_2 and m_0 depend on Λ and λ. We usually split the Z_2 factor into two $\sqrt{Z_2}$ factors, and associate each of the $\sqrt{Z_2}$ factors with one of the vertices. Then, the remaining electron propagator without Z_2 is finite

$$\begin{aligned} i\tilde{S}_F(p) &= Z_2^{-1}\, iS'_F(p) \\ &= \frac{i}{(\not{p} - m)(1 + C(\not{p}) + O(\alpha^2))}. \end{aligned} \tag{6.27}$$

The divergence of Z_2 is very mild. We have seen in e_0 how mild the Λ-dependence is. The same is also true for the infrared divergence of Z_2. If we choose λ to be the energy associated with a photon whose wave length equals the radius of the earth, the correction to m is only 10%. The presence of infrared divergences is caused by our asking inappropriate questions. As we have seen earlier, the emission of soft photons are unavoidable even in the semiclassical limit. These emissions are reflected as infrared divergences. If we ask about the inclusive cross sections with the presence of an indefinite number of photons up to a certain amount of energy; then, the infrared divergence disappear.

4. Vertex Corrections

The remaining class of the divergent diagrams is the vertex correction. The lowest order vertex correction is shown in Fig. 6.8.

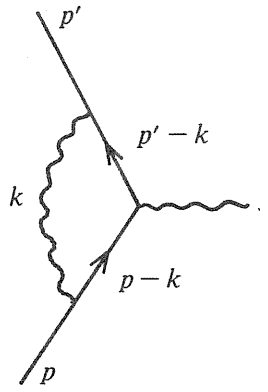


Fig. 6.8 The lowest order vertex correction.

The amplitude of this diagram is

$$\Lambda_\mu = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2-\lambda^2+i\varepsilon} \gamma_\nu$$

$$\times \frac{i}{\not p'-\not k-m+i\varepsilon} \gamma_\mu \frac{i}{\not p-\not k-m+i\varepsilon} \gamma_\nu .$$

$$= -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\nu(\not p'-\not k+m)\,\gamma_\mu\,(\not p-\not k+m)\,\gamma_\nu}{(k^2-\lambda^2+i\varepsilon)\,((p'-k)^2-m^2+i\varepsilon)\,((p-k)^2-m^2+i\varepsilon)} . \tag{6.28}$$

This diagram has a logarithmic ultraviolet divergence. To see the ultraviolet divergence (U.D.), we consider the integration region $m << |k| < \Lambda$, giving

$$(\Lambda_\mu)_{\mathrm{UD}} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma_\nu \not k\, \gamma_\mu \not k\, \gamma_\nu}{(k^2)^3}$$

$$= -ie^2 \int_{\Lambda > |k| >> m^2} \frac{d^4k}{(2\pi)^4} \frac{k^2\gamma_\mu}{(k^2)^3}$$

$$= \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} \gamma_\mu$$

where Λ is the ultraviolet cutoff. It is important to see that the ultraviolet divergent part is proportional to γ_μ. Hence, we can write

$$\Lambda_\mu(p', p) = \Lambda_\mu(p, p)\big|_{\not p = m} + \tilde{\Lambda}_\mu(p', p)$$

$$= (Z_1^{-1} - 1)\,\gamma_\mu + \tilde{\Lambda}_\mu(p', p) \tag{6.29}$$

where $Z_1^{-1} - 1$ is a diverget constant calculated at $P' = p$, $\not p = m$, and $\tilde{\Lambda}_\mu(p', p)$ is ultraviolet finite to $O(\alpha)$. From the above calculation, we find

$$Z_1^{-1} - 1 = \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2}$$

or

$$Z_1^{-1} = 1 + \frac{\alpha}{4\pi} \ln \frac{\Lambda^2}{m^2} + \ldots = Z_2^{-1} . \tag{6.30}$$

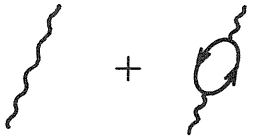
Actually, we can prove that $Z_1 = Z_2$ to all orders of α if we choose the finite part $\tilde{\Lambda}_\mu$ to satisfy $\bar{u}\tilde{\Lambda}_\mu u = 0$ at $p' = p$, $\not{p} = m$. This relation is a consequence of the Ward identity.

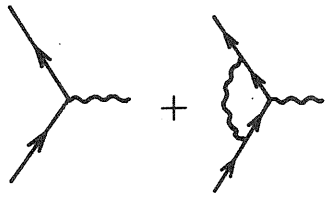
Using (6.29), we can work out the total vertex contribution as

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu + \Lambda_\mu \\ &= Z_1^{-1} (\gamma_\mu + \tilde{\Lambda}_\mu + O(\alpha^2)) \\ &= Z_1^{-1} \tilde{\Gamma}_\mu \end{aligned} \tag{6.31}$$

where $\tilde{\Lambda}_\mu$ and $\tilde{\Gamma}_\mu$ are finite and cutoff independent. It is important to note that, at least to $O(e^2)$, we can express the full propagators and the vertices as

$iS'_{\mathrm{F}} \equiv$ $= Z_2 \, i\tilde{S}_{\mathrm{F}}$ (6.32a)

$-i(D'_{\mathrm{F}})_{\mu\nu} =$  $= Z_3 (-i\tilde{D}_{\mathrm{F}\mu\nu})$ (6.32b)

$\Gamma_\mu =$  $= Z_1^{-1} \tilde{\Gamma}_\mu$ (6.32c)

where $Z's$ are cutoff-dependent constants, and $i\tilde{S}_{\mathrm{F}}$, $-i\tilde{D}_{\mathrm{F}\mu\nu}$ and $\tilde{\Gamma}_\mu$ are finite quantities. In a higher order calculation, we can combine these cutoff dependent constants with e_0 to give a physical measurable e. It is obvious that we have a Z_1^{-1} for each vertex. For Z_2 and Z_3 appearing in the electron and the photon propagators, we can associate $\sqrt{Z_2}$ for each of the fermion lines, and $\sqrt{Z_3}$ for each of the photon lines (See Fig. 6.9)

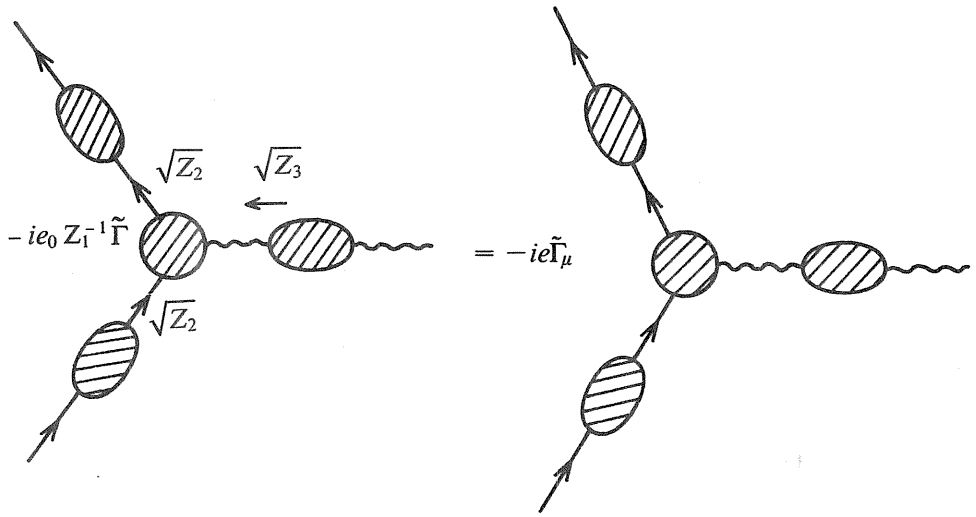


Fig. 6.9 A physical vertex.

At each vertex, the combination of these factors lead to

$$e = Z_2 \sqrt{Z_3}\, Z_1^{-1}\, e_0, \tag{6.33}$$

which, by the use of the Ward identity, reduces to

$$e = \sqrt{Z_3}\, e_0$$

as mentioned earlier.

We shall compute the finite part $\tilde{\Lambda}_\mu$ (or $\tilde{\Gamma}_\mu$) to $O(\alpha)$. For simplicity, we concentrate on the simple case of $\not{p} = m$, $\not{p}' = m$, but $\not{p}'_\mu - p_\mu \neq 0$. From (6.28), we have

$$\bar{u}(p')\,\Lambda_\mu\, u(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4}$$

$$\times \frac{\bar{u}(p')\,\gamma_\nu\,(\not{p}'-\not{k}+m)\,\gamma_\mu\,(\not{p}-\not{k}+m)\,\gamma_\nu\,u(p)}{(k^2-\lambda^2+i\varepsilon)\,((p'-k)^2-m^2+i\varepsilon)\,((p-k)^2-m^2+i\varepsilon)}. \tag{6.34}$$

For the numerator N_μ, we encounter

$$(\not{p}-\not{k}+m)\,\gamma_\nu\,u(p) = (2p_\nu - \not{k}\gamma_\nu)\,u(p) - \gamma_\nu \cancelto{0}{(\not{p}-m)\,u(p)}$$

$$\bar{u}(p')\,\gamma_\nu\,(\not{p}'-\not{k}+m) = \bar{u}(p')\,(2p'_\nu - \gamma_\nu \not{k}) + 0$$

and hence

$$
\begin{aligned}
N_\mu &\equiv \bar{u}(p')\,\gamma_\nu(\not{p}'-\not{k}+m)\,\gamma_\mu(\not{p}-\not{k}+m)\,\gamma_\nu u(p) \\
&= \bar{u}(p')\,(2p'_\nu-\gamma_\nu\not{k})\,\gamma_\mu(2p_\nu-\not{k}\gamma_\nu)\,u(p) \\
&= 4\,p'\cdot p\,\bar{u}\,\gamma_\mu u - 2\,\bar{u}(p')\,(\gamma_\mu\not{k}\not{p}'+\not{p}\not{k}\,\gamma_\mu)\,u(p) \\
&\quad + \bar{u}(p')\,\gamma_\nu\not{k}\,\gamma_\mu\not{k}\,\gamma_\nu u(p) \\
&= 4\,(p'\cdot p - k\cdot p' - k\cdot p + k^2/2)\,\bar{u}\,\gamma_\mu u \\
&\quad - 4\,mk_\mu\,\bar{u}u + 4\,(p_\mu+p'_\mu-k_\mu)\,\bar{u}\,\not{k}u. \qquad (6.35)
\end{aligned}
$$

To simplify the denominator, we use the Feynman parameter representation,

$$
\frac{1}{ABC} = 2!\int \frac{d\alpha_1\,d\alpha_2\,d\alpha_3\,\delta(\alpha_1+\alpha_2+\alpha_3-1)}{(\alpha_1 A+\alpha_2 B+\alpha_3 C)^3}
$$

and obtain, after the change of variables,

$$
\alpha_1 = xy, \quad \alpha_2 = x(1-y), \quad \alpha_3 = 1-x,
$$

$$
\frac{1}{ABC} = 2\int_0^1 x\,dx\int_0^1 dy\,\frac{1}{[xyA+x(1-y)B+(1-x)C]^3}. \qquad (6.36)
$$

Then, we can express the denominator as

$$
\frac{1}{((p-k)^2-m^2+i\varepsilon)\,((p'-k)^2-m^2+i\varepsilon)\,(k^2-\lambda^2+i\varepsilon)}
$$

$$
= 2\int_0^1 xdx\int_0^1 dy\,\frac{1}{[xy((p-k)^2-m^2)+x(1-y)((p'-k)^2-m^2)+(1-x)\,(k^2-\lambda^2)+i\varepsilon]^3}
$$

$$
= 2\int_0^1 xdx\int_0^1 dy\,\frac{1}{[k'^2-x^2m^2+x^2y(1-y)q^2-(1-x)\lambda^2+i\varepsilon]^3}
$$

with

$$k'_\mu = k_\mu - xyp_\mu - x(1-y)\,p'_\mu,$$

$$q^2 = (p'-p)^2.$$

With an appropriate cutoff, we are ready to do the k'-integration. We then express N_μ in terms of k', ignore terms linear in k', and replace $k'_\mu k'_\nu$ by $\frac{1}{4}g_{\mu\nu}k'^2$. This leads to

$$\begin{aligned}
N_\mu \to 4\,[&p'\cdot p - xy\,p'\cdot p - x(1-y)p'^2 - xyp^2 \\
&- x(1-y)\,p'\cdot p + \frac{k'^2}{2} + (xyp + x(1-y)p')^2\Big]\,\bar{u}\gamma_\mu u \\
&- 4mx\,(yp_\mu + (1-y)\,p'_\mu)\,\bar{u}u \\
&+ 4\,[p_\mu + p'_\mu - (xyp_\mu + x(1-y)p'_\mu)]\,\bar{u}(xy\not{p} + x(1-y)\not{p}')u \\
&- 4\,k'_\mu\,\bar{u}\,\not{k}'u \\
= 4\,[&p'\cdot p - xp'\cdot p - xm^2 - k'^2/2 + x^2m^2\,(y^2+(1-y)^2) \\
&+ 2x^2\,y(1-y)\,p'\cdot p]\,\bar{u}\,\gamma_\mu u \\
&+ 4xm\,[(1-y-xy)\,p_\mu + (y-x(1-y))p'_\mu]\bar{u}u\,.
\end{aligned}$$

Since the denominator is symmetric with respect to $y \leftrightarrow (1-y)$, we can average N_μ over y and $(1-y)$, giving

$$\begin{aligned}
N_\mu \to 4\,[&(1 - x + 2x^2y\,(1-y))\,p'\cdot p \\
&- xm^2\,(1 - x(y^2+(1-y)^2)) - k'^2/2]\,\bar{u}\,\gamma_\mu u \\
&+ 4xm\left[\left(-\frac{1}{2} - \frac{x}{2}\right)p_\mu + \left(\frac{1}{2} - \frac{x}{2}\right)p'_\mu\right]\bar{u}u \\
= 4\,[&\cdots]\,\bar{u}\,\gamma_\mu u \\
&+ 2xm\,(1-x)\,(p_\mu + p'_\mu)\,\bar{u}u.
\end{aligned}$$

The three conserved tensors $\bar{u}\gamma_\mu u$, $(p+p')_\mu \bar{u}u$, and $\bar{u}\sigma_{\mu\nu}uq^\nu$ are not independent. They are related by a relation known as the Gordon decomposition relation,

$$(p+p')_\mu \bar{u}u = 2m\bar{u}\gamma_\mu u - i\bar{u}\sigma_{\mu\nu}uq^\nu .$$

Using this relation, we eliminate $(p_\mu+p'_\mu)\bar{u}u$ in N_μ in favour of $\bar{u}\sigma_{\mu\nu}uq_\nu$, giving

$$N_\mu \to [4(1-x)m^2 - 2(1-x+2x^2y(1-y))\,q^2 - 2k'^2]\,\bar{u}\gamma_\mu u$$

$$-i\,2x(1-x)\,m\,\bar{u}\,\sigma_{\mu\nu}uq_\nu .$$

We can carry out the k'-integration to give

$$\Lambda_\mu = -ie^2\,2\int_0^1 xdxdy \int \frac{d^4k'}{(2\pi)^4}$$

$$\times \frac{[4(1-x)m^2 - 2(1-x+2x^2y(1-y))q^2 - 2k'^2]\bar{u}\gamma_\mu u - i2x(1-x)m\,\bar{u}\sigma_{\mu\nu}uq_\nu}{[k'^2 - x^2m^2 + x^2y(1-y)q^2 - (1-x)\lambda^2 + i\varepsilon]^3}$$

$$= -\frac{\alpha}{2\pi}\int_0^1 xdx \int_0^1 dy$$

$$\times \left\{\frac{[4(1-x)m^2 - 2(1-x+2x^2y(1-y))q^2]\bar{u}\gamma_\mu u - i2x(1-x)m\,\bar{u}\sigma_{\mu\nu}uq_\nu}{2(x^2m^2 - x^2y(1-y) - q^2 + (1-x)\lambda^2)}\right.$$

$$\left. + 2\bar{u}\gamma_\mu u\left(\ln\frac{m^2}{x^2m^2 - x^2y(1-y)q^2 + (1-x)\lambda^2} - \frac{3}{2}\right)\right\}$$

and

$$\tilde{\Lambda}_\mu(p', p) = \Lambda_\mu(p', p) - \Lambda_\mu(p, p)|_{\not{p}=m}.$$

$$= -\frac{\alpha}{2\pi}\int_0^1 xdx \int dy \left\{\left[\frac{2(1-x)m^2 - (1-x+2x^2y(1-y))q^2}{x^2m^2 - x^2y(1-y)q^2 + (1-x)\lambda^2}\right.\right.$$

$$- \frac{2(1-x)m^2}{x^2m^2+(1-x)\lambda^2} + \ln\frac{m^2}{m^2-y(1-y)q^2}\Bigg]\bar{u}(p')\gamma_\mu u(p)$$

$$- \frac{i(1-x)\,\bar{u}\sigma_{\mu\nu}uq_\nu/m}{x(1-y(1-y)q^2/m^2)}\Bigg\}.$$

We can express the total vertex contribution as

$$\tilde{\Gamma}_\mu = \gamma_\mu + \tilde{\Lambda}_\mu,$$

$$\bar{u}\tilde{\Gamma}_\mu u = \bar{u}(p')\left[\gamma_\mu F_1(q^2) + \frac{i\sigma_{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p)$$

where

$$F_1(q^2) = 1 - \frac{\alpha}{2\pi}\int_0^1 x\,dx\int_0^1 dy$$

$$\times\left[\frac{2(1-x)m^2-(1-x+2x^2y(1-y))q^2}{x^2m^2-x^2y(1-y)q^2+(1-x)\lambda^2} - \frac{2(1-x)m^2}{x^2m^2+(1-x)\lambda^2}\right.$$

$$\left. + \ln\frac{m^2}{m^2-y(1-y)q^2}\right],$$

$$F_2(q^2) = \frac{\alpha}{\pi}\int_0^1 dx\int_0^1 dy\,\frac{(1-x)}{1-y(1-y)q^2/m^2}$$

$$= \frac{\alpha}{2\pi}\int_0^1 dy\,\frac{1}{1-y(1-y)q^2/m^2}.$$

Expressions F_1 and F_2 are the electric and the magnetic form factors of the electron. The total electric charge is eF_1 (0), which is e. $F_2(0)$ gives an additional contribution to the magnetic moment,

$$F_2(0) = \frac{\alpha}{2\pi}\left(= \frac{g-2}{2}\right).$$

To $O(\alpha)$, the total magnetic moment of an electron is

$$\mu = 1 + \alpha/2\pi.$$

The quantity $\alpha/2\pi$ is known as the Schwinger correction. The recent experimental results for the anomalous magentic moments of $e^{\pm}$ and $\mu^{\pm}$ are (see, T. Kinoshita *et al.*, *Phys. Rev. Lett.* **52** (1984) 717, **47** (1981) 1573)

$$(\mu-1)^{e^-}_{\text{exp}} = 1159652200\,(\pm 40) \times 10^{-12}$$

$$(\mu-1)^{e^+}_{\text{exp}} = 1159652222\,(\pm 50) \times 10^{-12}$$

$$(\mu-1)^{\mu^-}_{\text{exp}} = 11659370\,(\pm 120) \times 10^{-10}$$

$$(\mu-1)^{\mu^+}_{\text{exp}} = 11659110\,(\pm 110) \times 10^{-10}$$

$$\alpha^{-1}_{\text{exp}} = 137.035963\,(\pm 13) \quad \text{(by Josephson Junction)}$$

and the theoretical results are

$$(\mu-1)^{e^{\pm}}_{\text{Th}} = 0.5\left(\frac{\alpha}{2\pi}\right) - 0.328478966\left(\frac{\alpha}{\pi}\right)^2$$

$$+ 1.1765\,(\pm 13)\left(\frac{\alpha}{\pi}\right)^3 + (-0.8 \pm 2.5)\left(\frac{\alpha}{\pi}\right)^4$$

$$= 1159652460\,(\underset{\text{due to } \alpha}{\pm\ 127}\ \underset{\text{Theory}}{\pm\ 75}) \times 10^{-12}$$

$$(\mu-1)^{\mu^{\pm}}_{\text{Th}} = 11659202\,(\pm 20) \times 10^{-10}.$$

For the anomalous magnetic moment of $e^{\pm}$, the agreement between the theory and the experiment is about 3 parts in 10^{10}! The uncertainty is mainly in the determination of the hyperfine structure constant α.

Chapter 7

ZERO TEMPERATURE FERMI SYSTEM

1. Introduction

In this Chapter, we consider a nonrelativistic zero temperature fermi system. The fermions may be either electrons or nucleons. We assume that the interactions are instantaneous. In the nonrelativistic limit, a typical Hamiltonian is

$$H = H_0 + H_{\text{int}} \tag{7.1}$$

with

$$H_0 = \int d^3x\, \psi_\alpha^\dagger(x) \left(\frac{-\delta_{\alpha\alpha'}}{2m} \nabla^2 + U_{\alpha\alpha'}(x) \right) \psi_{\alpha'}(x) \tag{7.2}$$

$$H_{\text{int}} = \frac{1}{2} \int d^3x\, d^3x'\, \psi_\alpha^\dagger(x)\, \psi_\beta^\dagger(x')\, V(x, x')_{\alpha\alpha', \beta\beta'}\, \psi_{\beta'}(x')\, \psi_{\alpha'}(x). \tag{7.3}$$

In Eqs. (7.1)–(7.3), there are only one and two-body potentials. In the nonrelativistic limit, the spinor field variables ψ and $\psi^\dagger$ contain only two nontrivial components. The kinetic term is spin-independent. $U_{\alpha\alpha'}$ is a one-particle potential, and $V_{\alpha\alpha', \beta\beta'}$ is a two-particle potential. Both potentials may be spin-dependent. The field variables in H_{int} is normal-ordered according to the free particle creation and annihilation operators defined in the empty space.

In many applications, the ground state of interest is not the ground state defined in the empty space. We are interested in the ground state where all the one-particle states up to a fermi energy ε_F are occupied, and all the one-particle states for energy larger than ε_F are empty. This is our new ground state. The creation and the annihilation operators of the empty space ground state, the vacuum, are not necessarily the creation and the annihilation operators of the new ground state. The operators which are normal-ordered according to the vacuum state are not necessarily normal-ordered according to the new creation and annihilation operators.

2. The Feynman Rules

We begin with the Hamiltonian.

$$H_0 = \int d^3x\, \psi^\dagger(x) \left(-\frac{1}{2m} \nabla^2 + U(x) \right) \psi(x). \tag{7.4}$$

The ψ and $\psi^\dagger$ obey the canonical equal-time quantization relations,

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0 \tag{7.5a}$$

$$\{\psi_\alpha(x, t), \psi^\dagger_\beta(y, t)\} = \delta_{\alpha\beta}\, \delta^3(x-y). \tag{7.5b}$$

The Heisenberg equations of motion are

$$i\dot{\psi} = [\psi, H_0] = \left(-\frac{1}{2m} \nabla^2 + U(x) \right) \psi \tag{7.6a}$$

$$i\dot{\psi}^\dagger = [\psi^\dagger, H_0] = -\psi^\dagger \left(-\frac{1}{2m} \overleftarrow{\nabla}^2 + U(x) \right). \tag{7.6b}$$

Equation (7.6) has the form of a one-particle Schrödinger equation.

For simplicity, we consider the case of $U = 0$. We can construct the plane wave solution $u(s)$ exp $(-iEt + \mathbf{p}\cdot\mathbf{x})$ where $E = p^2/2m$. We can express ψ and $\psi^\dagger$ as

$$\psi_\alpha(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}}\, u(s)_\alpha\, e^{-iEt + i\mathbf{p}\cdot\mathbf{x}}\, c(p, s) \tag{7.7a}$$

$$\psi^\dagger_\alpha(x) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3}}\, u^\dagger(s)_\alpha\, e^{iEt - i\mathbf{p}\cdot\mathbf{x}}\, c^\dagger(p, s) \tag{7.7b}$$

where $u(s)$ is a 2-component spinor obeying

$$u^\dagger(s')\, u(s) = \delta_{s's}, \qquad s, s' = 1, 2 \tag{7.8a}$$

$$\sum_s u(s)\, u^\dagger(s) = 1, \qquad 2\times 2 \text{ matrix}. \tag{7.8b}$$

In the nonrelativistic limit, we may choose $u(s)$ to be independent of $\mathbf{p}$. We can interpret, as before, the coefficients c and $c^\dagger$ as annihilation and creation operators,

$$\{c, c\} = \{c^\dagger, c^\dagger\} = 0 \tag{7.9a}$$

$$\{c(p, s), c^{\dagger}(p', s')\} = \delta^3(p - p')\,\delta_{ss'}\,. \tag{7.9b}$$

We can also recover Eqs. (7.7)–(7.9) directly from the analogous Dirac expressions by ignoring the antiparticle solutions and by factoring out a constant phase exp(-*imt*) in ψ. The vacuum state, $|\,0\,\rangle$, satisifes

$$c(p, s)\,|\,0\,\rangle = 0 \qquad \text{for all } p\text{'s} \ldots . \tag{7.10}$$

For the free fermion system in the presence of a fermi sea with the fermi energy ε_F, we introduce a new ground state $|\,\Phi_0\,\rangle$. In Φ_0, all particle states with $E < \varepsilon_F$ are occupied, and all particle states with $E > \varepsilon_F$ are empty, giving

$$c(p, s)\,|\,\Phi_0\,\rangle = 0, \quad E > \varepsilon_F \tag{7.11a}$$

$$c^{\dagger}(p, s)\,|\,\Phi_0\,\rangle = 0, \quad E < \varepsilon_F\,. \tag{7.11b}$$

It is now natural to introduce the annihilation and creation operators for the particles ($E > \varepsilon_F$) and the holes ($0 < E < \varepsilon_F$) as

$$\left.\begin{aligned} a(p, s) &= c(p, s) \\ a^{\dagger}(p, s) &= c^{\dagger}(p, s) \end{aligned}\right\} E > \varepsilon_F \tag{7.12a}$$

$$\left.\begin{aligned} b(p, s) &= c^{\dagger}(p, s) \\ b^{\dagger}(p, s) &= c(p, s) \end{aligned}\right\} 0 < E < \varepsilon_F\,. \tag{7.12b}$$

We then introduce the normal-ordered products according to the order of a's and b's. We can introduce the particle/hole propagation function $iG(x, x')$ by

$$\begin{aligned} iG(x, x')_{\alpha\beta} &= \overbrace{\psi_{\alpha}(x)\,\psi^{\dagger}_{\beta}(x')} \\ &\equiv T(\psi_{\alpha}(x)\,\psi^{\dagger}_{\beta}(x')) - :\psi_{\alpha}(x)\,\psi^{\dagger}_{\beta}(x'):\\ &= \delta_{\alpha\beta}\int \frac{d^3p}{(2\pi)^3}[e^{-iE_p(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\,\theta(p-p_F)\,\theta(t-t') \\ &\quad - e^{-iE_p(t'-t)+i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})}\,\theta(p_F-p)\,\theta(t'-t)] \end{aligned} \tag{7.13}$$

where $E_p = p^2/2m$ is the on-shell energy, and $p_F = \sqrt{2m\varepsilon_F}$ is the fermi momentum. In (7.13), we may interpret the "holes" as particles propagating backward in time.

We can express (7.13) as a four-dimensional fourier representation,

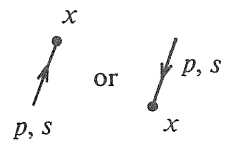
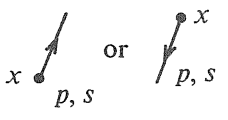
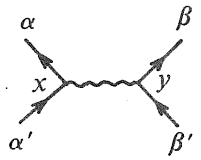
$$iG(x, x')_{\alpha\beta} = \int \frac{d^4p}{(2\pi)^4} e^{-ip_0(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} \times \frac{i\delta_{\alpha\beta}}{p_0 - \mathbf{p}^2/2m + i\varepsilon\,\mathrm{SGN}(p-p_F)} \tag{7.14}$$

where

$$\mathrm{SGN}(p-p_F) = \begin{cases} +1 & \text{for } p > p_F \\ -1 & \text{for } p < p_F \end{cases}. \tag{7.15}$$

We can derive the Feynman rules through the same Dyson-Wick procedure: We introduce the interaction picture, time-ordered products, etc. We summarize the position-space and the momentum-space Feynman rules in the next two tables. In Table III, we present the position space Feynman amplitudes. We need to integrate over all internal

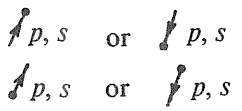
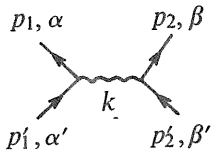
Table III. Position space Feynman rules for zero temperature fermi systems.

Process	Graphical Representation	Amplitude	Remark
External Particles		$\frac{1}{\sqrt{(2\pi)^3}} e^{i\mathbf{p}\cdot\mathbf{x}-iEt} u(s)$	Replace $\frac{1}{\sqrt{(2\pi)^3}}$ by $\frac{1}{\sqrt{V}}$ for a finite box normalization
		$\frac{1}{\sqrt{(2\pi)^3}} e^{-i\mathbf{p}\cdot\mathbf{x}+iEt} u^*(s)$	
Internal lines		$iG(x, y)_{\alpha\beta}$	See Eqs. (7.13), (7.14)
Interaction		$U(x, y)_{\alpha\alpha',\beta\beta'} \equiv V(x, y)_{\alpha\alpha',\beta\beta'} \times \delta(x_0-y_0)$	instantaneous interaction

Fermion loop		$(-1)\,\mathrm{Tr}\,(\ldots)$	
nth order		$\frac{(-i)^n}{n!}\int d^4x$	Sum over permutation turns $\frac{1}{n!}\to\frac{1}{S}$

coordinates, and sum over n. The factor $1/n!$ is partially cancelled when we include the contributions associated with the permutations of the vertices. Only a symmetry factor $1/S$ is remaining. Note that lowest order interaction U(or V) corresponds to g^2 order in the particle-physics interaction. In Table IV, we present the Feynman rules for the invariant amplitudes in the momentum space.

Table IV. Momentum space Feynman rules for zero temperature fermi systems.

Process	Graphical Representation	Amplitude	Remark
External particles		$u(s)$ $u^*(s)$	
Internal lines		$\frac{i\delta_{\alpha\beta}}{p_0-p^2/2m+i\varepsilon\mathrm{SGN}(p-p_F)}$	$\mathrm{SGN}(p-p_F)$ $= \begin{matrix}1 & \text{for } p>p_F\\ -1 & p<p_F\end{matrix}$
Interaction		$U(\mathbf{k})_{\alpha\alpha',\beta\beta'}$ $\times (2\pi)^4\delta^4(p_1'+p_2'-p_1-p_2).$	$k=p_1'-p_1$, $U(k)$ is the Fourier transform of $U(x-y)$
Fermion loop		$(-1)\,\mathrm{Tr}\,(\ldots)$	
nth order		$\frac{(-i)^n}{n!}\int\frac{d^4p}{(2\pi)^4}$	Sum over permutation turns $\frac{1}{n!}\to\frac{1}{s}$

3. Hartree-Fock Approximation

The presence of a fermi level indicates that even to the zeroth order of U, there is a shift of the ground state energy density,

$$\frac{(\Delta E)^0}{V} = g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{2m} \theta(p_F - p)$$

$$= g\, p_F^5 / ((10\pi^2)\, 2m) \tag{7.16}$$

where $g = 2s+1$ is the multiplicity of the spin states. For spin-1/2 particles, $g = 2$. We can express the fermi momentum p_F in terms of the total fermion density N/V via

$$\frac{N}{V} = g \int \frac{d^3p}{(2\pi)^3} \theta(p_F - p)$$

$$= g\, p_F^3 / (6\pi^2). \tag{7.17}$$

Hence, we have the average fermi energy per particle as

$$\frac{(\Delta E)^0}{N} = \frac{3}{5} \frac{p_F^2}{2m} = \frac{3}{5} \varepsilon_F . \tag{7.18}$$

To the lowest order in U, we have two vacuum diagrams as shown in Fig. 7.1.

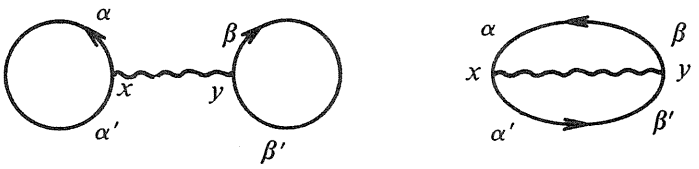


Fig. 7.1 Hartree-Fock vacuum diagrams.

We shall first study diagram (7.1a). Here, we encounter a tadpole term $iG(x, x)$. The expression (7.13) is ill-defined at $t' = t$ because the θ-functions are ambiguous. If we treat $\theta(0)$ as $1/2$, we encounter a divergent result. We need a careful assessment of the physical situation. We understand that Eq. (7.13) is the nonrelativistic limit of a more precise description. We have ignored in (7.13) the antiparticle contributions which all have a huge phase factor $\exp[-2im(t'-t)]$. As long as $t'-t \neq 0$,

this rapidly changing phase kills the amplitude as one varies $t'-t$. However, at $t' = t$, this phase becomes one and we can no longer ignore the antiparticle contributions. At $\varepsilon_F = 0$, the infinite constants due to the particles and the antiparticles cancel each other, as described by the Furry's Theorem.

For $\varepsilon_F \neq 0$, we have a finite result,

$$iG(x, x')_{\alpha\beta} = \delta_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} [e^{-iE_p(t-t')+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\theta(t'-t)\theta(p-p_F)$$

$$- e^{-iE_p(t'-t)+i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})}\theta(t-t')\theta(p_F-p)$$

$$- \theta(t-t') \times \text{antiparticle contribution}]$$

and, hence, at $x' = x$,

$$iG(x, x')_{\alpha\beta} = -\delta_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3} \theta(p_F-p)$$

$$= -\frac{\delta_{\alpha\beta}}{g}\frac{N}{V} = -n\delta_{\alpha\beta}/g \tag{7.19}$$

where n is the particle density. We can formally recover (7.19) from (7.13) if we interpret the tadpole term as

$$iG(x, x)_{\alpha\beta} = iG(x, x')\big|_{\mathbf{x}=\mathbf{x}',\, x_0'=x_0^+}. \tag{7.20}$$

The limit $x_0' = x_0^+$ is an artificial procedure to insure that only the particles in the fermi-sea contribute. We should follow the same procedure in evaluating diagram (7.1b) where we encounter $iG(x, y)$ at $x_0 = y_0$. Just as in the tadpole case, we can no longer ignore the antiparticle contributions since the phase factor $\exp[2im(x_0 - y_0)]$ becomes one again at $x_0 = y_0$. At $\varepsilon_F = 0$, the particle and the antiparticle contributions give rise to the observed energy of the physical vacuum. The additional contribution to diagram (7.1b) can only come from the particles in the fermi sea. We then have

$$-i\Delta\varepsilon VT = A_a + A_b$$

$$= \frac{(-i)}{2}\int d^4x\, d^4y \{(-1)iG(x,x)_{\alpha'\alpha}$$

$$\times (-1)\, iG(y,y)_{\beta'\beta}\, U(x,y)_{\alpha\alpha',\beta\beta'}$$

$$+ (-)\, iG(x,y)_{\alpha\beta}\, iG(y,x)_{\beta'\alpha'}\, U(x,y)_{\alpha'\alpha,\beta\beta'}\} \qquad (7.21)$$

where $\Delta\varepsilon$ is the change of the ground state energy density due to Fig. 7.1. The first-term in (7.21) corresponds to Fig. 7.1a and the second term corresponds to Fig. 7.1b. Note that $G(x,x)$ and $G(y,y)$ are constants and hence the integrand of (7.21) depends only on $x-y$. Thus we can reduce the double integration to a VT factor times a single integral,

$$\int d^4x\, d^4y\, [\cdots] = VT \int d^4(x-y)\, [\cdots].$$

Cancelling the factor $-iVT$, we can express the change of the ground state energy density, $\Delta\varepsilon$, as

$$\Delta\varepsilon = \frac{1}{2}\int d^4(x-y)\, \{iG(x,x)_{\alpha\alpha'}\, iG(y,y)_{\beta'\beta}\, U(x,y)_{\alpha\alpha',\beta\beta'}$$

$$-iG(x,y)_{\alpha\beta}\, iG(y,x)_{\beta'\alpha'}\, U(x,y)_{\alpha'\alpha,\beta\beta'}\}$$

$$= \frac{1}{2}\int_{p<p_F} \frac{d^3p}{(2\pi)^3} \int_{p'<p_F} \frac{d^3p'}{(2\pi)^3}$$

$$\times [U_{\alpha\alpha,\beta\beta}(0) - U_{\alpha\beta,\beta\alpha}(p'-p)]. \qquad (7.22)$$

The first term is the change of the ground state energy due to the interaction of the fermions in the fermi sea, and the second term is required by the fermi statistics. The combined result is known as the Hartree–Fock approximation.

In addition to the ground state energy, we can also work out the self-energy and other higher order processes. For the lowest order self-energy diagrams, we have a similar Hartree–Fock contribution as shown in Fig. 7.2. The procedure that we have used to evaluate the Hartee–Fock

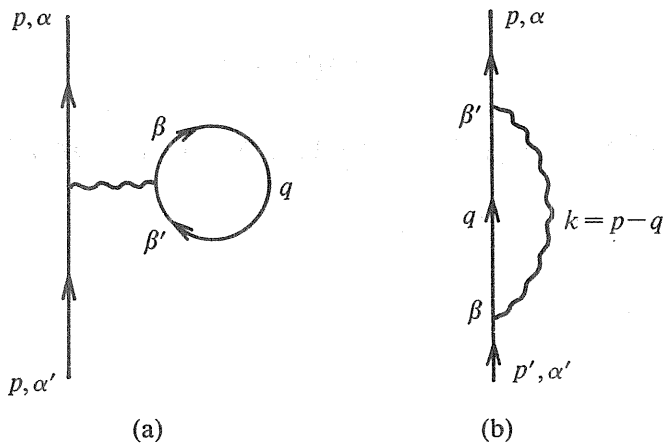


Fig. 7.2 Hartree-Fock self-energy diagrams.

ground state energy contribution also applies here. The proper self-energy for Fig. 7.2a and 7.2b are

$$-i\Sigma_a = (-i)(-1)\int \frac{d^4q}{(2\pi)^4} \frac{i\delta_{\beta\beta'}}{q_0 - q^2/2m \pm i\varepsilon} U_{\alpha\alpha',\beta\beta'}(0) \tag{7.23a}$$

$$-i\Sigma_b = (-i)\int \frac{d^4q}{(2\pi)^4} \frac{i\delta_{\beta\beta'}}{q_0 - q^2/2m \pm i\varepsilon} U_{\alpha\beta',\beta\alpha'}(p-q)\,. \tag{7.23b}$$

Integrating out q_0, and keeping only the fermi sea contribution, we have

$$\Sigma \equiv \Sigma_a + \Sigma_b = \int_{q<p_F} \frac{d^3q}{(2\pi)^3} [U_{\alpha\alpha',\beta\beta}(0) - U_{\alpha\beta,\beta\alpha'}(p-q)]\,. \tag{7.24}$$

For a spin-independent potential such as in the elctron gas model to be described later,

$$U_{\alpha\alpha',\beta\beta'}(k) = U(k)\,\delta_{\alpha\alpha'}\,\delta_{\beta\beta'},$$

then, $\Sigma_{\alpha\alpha'}$ is always proportional to $\delta_{\alpha\alpha'}$ and no spin flips are possible. For a spin-dependent potential such as

$$U_{\alpha\alpha',\beta\beta'} = V_1\,\delta_{\alpha\alpha'}\,\delta_{\beta\beta'} + V_2\,(\sigma^i)_{\alpha\alpha'}\,(\sigma^i)_{\beta\beta'},$$

$\Sigma_{\alpha\alpha'}$ will be more complicated.

We can evaluate the higher order processes in a similar way.

Chapter 8

THE ELECTRON GAS MODEL

1. The Model

We consider a model describing the electrons inside a metal, known as the electron gas model. This model consists of a system of electrons interacting with each other through the instantaneous Coulomb force. We also include a uniform background of positive charge to cancel the infinite static Coulomb potential. This uniform positive background is an idealization of the positive metal ions distributed inside the metal. Note that a static uniform background of positive charges does not contribute to any change of energy and momentum of the electrons. We encounter three kinds of interactions:

(a) The electron-electron interaction

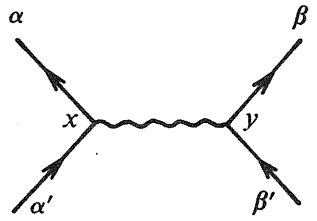


Fig. 8.1a

$$U_{\alpha\alpha',\beta\beta'}(x, y) = \frac{e^2 \delta_{\alpha\alpha'} \delta_{\beta\beta'}}{4\pi |\mathbf{x}-\mathbf{y}|}$$

$$U_{\alpha\alpha',\beta\beta'}(k) = \frac{e^2 \delta_{\alpha\alpha'} \delta_{\beta\beta'}}{|\mathbf{k}^2|}$$

(b) The electron-background interaction

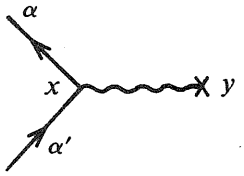


Fig. 8.1b

$$-\frac{e^2 \delta_{\alpha\alpha'} n}{4\pi |\mathbf{x}-\mathbf{y}|},$$

en = positive charge density

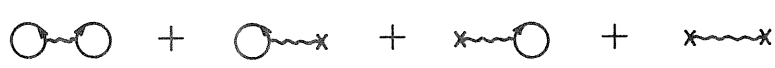
(c) The background-background interaction

Fig. 8.1c

$$\frac{e^2 nn}{4\pi |\mathbf{x}-\mathbf{y}|}$$

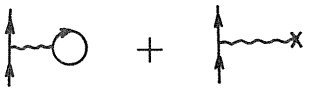
Note that we have kept the same definition of the electric charge as in the quantum electrodynamics. For this reason, we have $e^2/4\pi|\mathbf{x}-\mathbf{y}|$ as the Coulomb potential, instead of the usual $e^2/|\mathbf{x}-\mathbf{y}|$.

It is easy to see that the tadpole (Hartree) terms and the positive background terms cancel each other both in the ground state energy and in the self-energy contribution:



(8.1)

and



(8.2)

This cancellation occurs in higher order diagrams as well. Thus, in terms of Feynman diagrams, the only effect of the positive background is to cancel the tadpole (the Hartree) contribution.

2. Hartree–Fock Approximation

We can work out the Hartree–Fock contribution for the electron gas explicitly. For the ground state energy, only Fig. 7.1b remains, giving (See chap. 7, Eq. (7.22))

$$\begin{aligned}\Delta\varepsilon &= -\frac{1}{2}\int_{p<p_F}\frac{d^3p}{(2\pi)^3}\int\frac{d^3p'}{(2\pi)^3}U_{\alpha\beta,\beta\alpha}(p'-p)\\ &= -e^2\int_{p<p_F}\frac{d^3p}{(2\pi)^3}\int_{p'<p_F}\frac{d^3p'}{(2\pi)^3}\frac{1}{|p-p'|^2}.\end{aligned} \tag{8.3}$$

The evaluation of (8.3) is straightforward. We can obtain the essential p dependence from a simple scaling argument. In terms of

$$l = p/p_F \quad , \quad l' = p'/p_F \tag{8.4}$$

with

$$l, l' < 1$$

we have

$$\Delta\varepsilon = -e^2 p_F^4 \int_{l<1} \frac{d^3l}{(2\pi)^3} \int_{l'<1} \frac{d^3l'}{(2\pi)^3} \frac{1}{|l-l'|^2}$$

$$= -\text{const}\, e^2 p_F^4 \tag{8.5}$$

where the constant is $1/(16\pi^4)$. We can compare the above change of the ground state energy with the zeroth order fermi energy in Chap. 7, Eq. (7.16),

$$\varepsilon^0 = (2s+1)\, p_F^5/(20\pi^2 m). \tag{8.6}$$

At the high density limit, we find that $p_F >> e^2 m$ and consequently, the interaction energy is negligible.

Similarly, we can work out the Hartree–Fock self-energy of an electron gas. After the cancellation in Eq. (8.2), only Fig. 7.2b contributes, giving (see Chap. 7, Eq. (7.24))

$$\Sigma(p)_{\alpha\alpha'} = \int_{q<p_F} \frac{d^3q}{(2\pi)^3} U_{\alpha\beta,\beta\alpha'}(p-q)$$

$$= -\int_{q<p_F} \frac{d^3q}{(2\pi)^3} \frac{e^2 \delta_{\alpha\beta}\delta_{\beta\alpha'}}{|\mathbf{p}-\mathbf{q}|^2}$$

$$= -\frac{e^2 \delta_{\alpha\alpha'}}{(2\pi)^3} \int_0^{p_F} 2\pi q^2 dq \int_{-1}^{1} d(\cos\theta) \frac{1}{p^2+q^2-2pq\cos\theta}$$

$$= -\frac{e^2 \delta_{\alpha\alpha'}}{(2\pi)^2 p} \int_0^{p_F} q\, dq\, \ln\frac{(p+q)^2}{(p-q)^2}$$

$$= -\frac{e^2 \delta_{\alpha\alpha'}}{(2\pi)^2 p} \int_0^{p_F} q\, dq\, (\ln(p+q) - \ln|p-q|)$$

$$= -\frac{\alpha\delta_{\alpha\alpha'}}{2\pi}\left[\frac{p_F^2-p^2}{p}\ln\left|\frac{p_F+p}{p_F-p}\right| + 2p_F\right] \tag{8.7}$$

where $\alpha = e^2/4\pi$. Note that the integration always converges even if $p < p_F$.

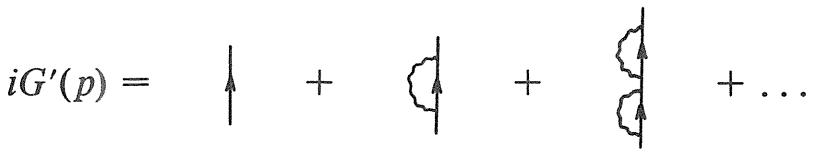


Fig. 8.2 Dyson equation.

The full propagator expressed in terms of the Dyson equation in Fig. 8.2 gives

$$iG'(p) = iG + iG(-i\Sigma)\, iG + iG(-i\Sigma)\, iG(-i\Sigma)\, iG + \ldots$$

$$= \frac{1}{1-G\Sigma}\, iG = \frac{i}{G^{-1}-\Sigma}$$

$$= \frac{i}{p_0 - \mathbf{p}^2/2m - \Sigma(p)}. \tag{8.8}$$

In general, $\Sigma(p)$ is complex. The imaginary part of Σ measures the decay width $(1/\tau)$ of the single particle excitation, and the real part of Σ denotes the modification of the energy-momentum relation. In the Hartree–Fock approximation, $\Sigma(p)$ is real and the single-particle state is stable.

We obtain the energy-momentum relation, known as the excitation-spectrum, in the Hartree–Fock approximation as

$$E(p) = p^2/2m + \Sigma(p)$$

$$= \frac{p^2}{2m} - \frac{\alpha}{2\pi}\left(\frac{p_F^2 - p^2}{p} \ln\left|\frac{p_F + p}{p_F - p}\right| + 2p_F\right). \tag{8.9}$$

We introduce the effective mass m^* near the fermi surface by

$$\left.\frac{\partial E(p)}{\partial p}\right|_{p_F} = \frac{p_F}{m^*}$$

or

$$m^* = p_F \Big/ \left(\frac{\partial E(p)}{\partial p}\right)_{p_F}. \tag{8.10}$$

From (8.32), we have

$$\frac{\partial E(p)}{\partial p} = \frac{p}{m} + \frac{\alpha}{2\pi}\left(\frac{p_F^2+p^2}{p}\ln\left|\frac{p_F+p}{p_F-p}\right| - \frac{2p_F}{p}\right)$$

$$\to \infty \qquad \text{as } p \to p_F . \tag{8.11}$$

In other words, $m^* = 0$ on the fermi-surface, and all metals are superconductors! Obviously, there is something wrong in our approximation. The trouble arises from the long range contribution of the Coulomb potential.

3. Polarization and Screening

The reason that we obtain an unphysical result $m^* = 0$ in the last section is due to the fact that we have not taken the screening into account. It is well-known that an electric field cannot penetrate into the interior of a metal. Thus, the effective interaction between the electrons is not given by the long range Coulomb force. We shall demonstrate that the ground state polarization gives rise to a screening potential.

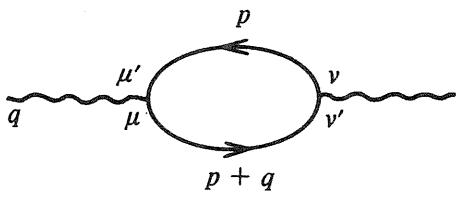


Fig. 8.3 Polarization diagram.

In Fig. 8.3, we introduce the lowest order polarization contribution as $\delta_{\mu'\nu}\delta_{\nu'\mu}\,\Pi(q)/2$ with

$$\Pi(q) = 2(-i)(-1)\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p_0-\mathbf{p}^2/2m+i\varepsilon\ \mathrm{SGN}(p-p_F)}$$

$$\times \frac{1}{(p_0+q_0)-(\mathbf{p}+\mathbf{q})^2/2m+i\varepsilon\ \mathrm{SGN}(|\mathbf{p}+\mathbf{q}|-p_F)} . \tag{8.12}$$

The sum of polarization diagrams in Fig. 8.4 gives rise to a screened potential. The corresponding Dyson Equation is

$$U'(q)_{\alpha\alpha',\beta\beta'} = \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\,[U(q) + U\Pi U + \cdots]$$

$$= \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\,\frac{U(q)}{1-\Pi U(q)}$$

$$= \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\,\frac{1}{U(q)^{-1}-\Pi(q)}$$

$$= \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\,\frac{e^2}{\mathbf{q}^2-e^2\Pi(q)}\,. \tag{8.13}$$

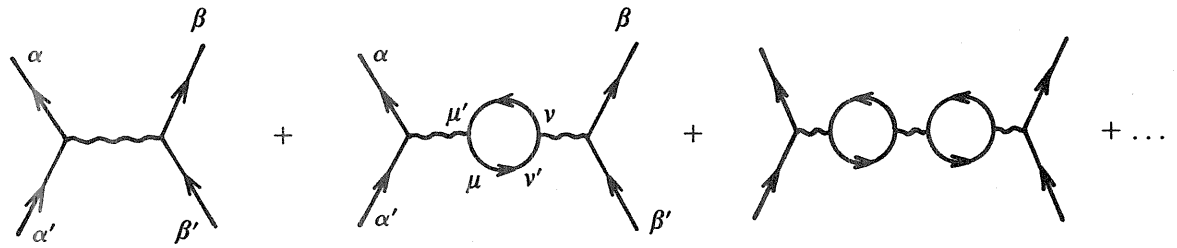


Fig. 8.4 The screened potential as a sum of polarization diagrams.

Note that $\Pi(q)$ is the analog of Π_{00} in QED. One may think that $\Pi(q)$ is proportional to q^2 at small q which insures a Coulomb potential at large distances. However, this is not true in the presence of the fermi-sea, and our $\Pi(q)$ actually becomes finite at $q=0$. The argument of Lorentz invariance which leads to

$$\Pi_{\mu\nu} = (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi(q^2)$$

in QED is no longer applicable here due to the presence of the fermi-sea.

We compute (8.12) in the static situation $q_0 = 0$, obtaining

$$\Pi(q) = 2i\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p_0 - \mathbf{p}^2/2m + i\varepsilon\ \mathrm{SGN}(p-P_F)}\cdot$$
$$\times\,\frac{i}{p_0 - (\mathbf{p}+\mathbf{q})^2/2m + i\varepsilon\,\mathrm{SGN}(|\mathbf{p}+\mathbf{q}| - P_F)}\,. \tag{8.14}$$

Equation (8.14) gives nonvanishing results only if the two poles of the de-

nominators are on the opposite sides of the real p_0 axis. This can happen in two situations: (1) $p < p_F$ but $|\mathbf{p}+\mathbf{q}| > p_F$ (Fig. 8.4a), (2) $p > p_F$ but $|\mathbf{p}+\mathbf{q}| > p_F$ (Fig. 8.4b).

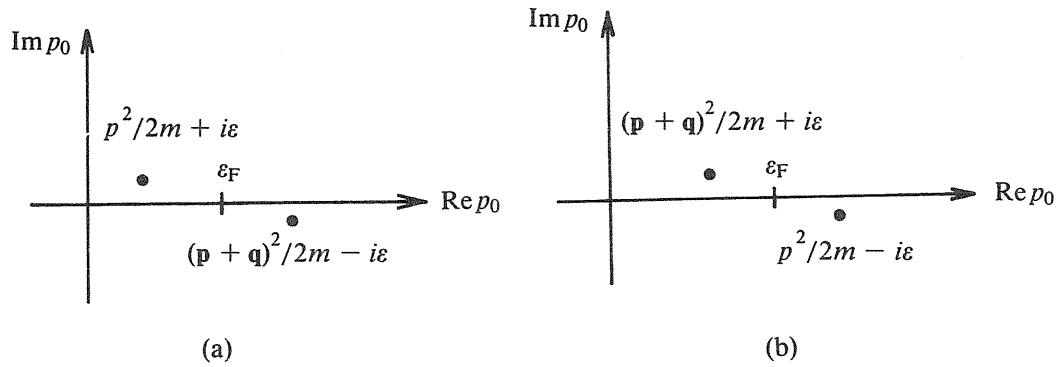


Fig. 8.5 The positions of poles in the complex p_0 plane. (a) $p < p_F, |\mathbf{p}+\mathbf{q}| > p_F$, and (b) $p > p_F, |\mathbf{p}+\mathbf{q}| < p_F$.

Then, we have

$$\Pi(q) = 2i \int \frac{d^3p}{(2\pi)^3} \int \frac{dp_0}{(2\pi)} \left\{ \frac{i\theta(p_F - p)}{p_0 - p^2/2m - i\varepsilon} \quad \frac{i\theta(|\mathbf{p}+\mathbf{q}| - p_F)}{p_0 - (\mathbf{p}+\mathbf{q})^2/2m + i\varepsilon} \right.$$

$$\left. + \frac{i\theta(p - p_F)}{p_0 - p^2/2m + i\varepsilon} \quad \frac{i\theta(p_F - |\mathbf{p}+\mathbf{q}|)}{p_0 - (\mathbf{p}+\mathbf{q})^2/2m - i\varepsilon} \right\}$$

$$= -2 \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\theta(p_F - p)\,\theta(|\mathbf{p}+\mathbf{q}| - p_F)}{(\mathbf{p}+\mathbf{q})^2/2m - p^2/2m} \right.$$

$$\left. + \frac{\theta(p_F - |\mathbf{p}+\mathbf{q}|)\,\theta(p - p_F)}{-(\mathbf{p}+\mathbf{q})^2/2m + p^2/2m} \right\}. \tag{8.15}$$

It is easy to see that $\Pi(q)$ is real (recall that $q_0 = 0$) and that the two terms in (8.15) contribute equally. The integral is elementary, giving

$$\Pi(q) = -\frac{mp_F}{2\pi^2} \left[1 - \frac{p_F}{q} \left(1 - \frac{q^2}{4p_F^2} \right) \ln \left| \frac{p_F - q/2}{p_F + q/2} \right| \right]. \tag{8.16}$$

In particular, as $q \to 0$, we have

$$\Pi(0) = -\frac{mp_F}{2\pi^2}\left[1 - \frac{p_F}{q}\left(-\frac{q}{p_F}\right)\right]$$

$$= -mp_F/\pi^2 . \tag{8.17}$$

We shall ignore the q^2 terms in $\Pi(q)$ because they contribute mainly to a dielectric effect. As we shall see, the non-vanishing term is far more significant. We can write (8.13) as

$$U'(q)_{\alpha\alpha',\beta\beta'} = \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\frac{e^2}{q^2 - e^2\,\Pi(0)}$$

$$= \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\frac{e^2}{\mathfrak{q}^2 + \lambda^2} \tag{8.18}$$

where

$$\lambda^2 \equiv -e^2\,\Pi(0) = \frac{e^2 mp_F}{\pi^2} = \frac{4\alpha mp_F}{\pi}. \tag{8.19}$$

In the position space, U' in (8.18) gives rise to a Yukawa potential

$$U'(r)_{\alpha\alpha',\beta\beta'} = \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\frac{e^2}{4\pi r}e^{-\lambda r}. \tag{8.20}$$

The electrons inside an electron gas see a screened Coulomb potential. With a screened potential, the Hartree–Fock calculation leads to a nonvanishing m^*. The metals are no longer superconductors.

We can understand the above screening from a simple physical approximation known as the Thomas–Fermi theory. Consider a point charge located at the origin. This charge, with its polarization, produces an electric potential $\phi(x)$ $(=A_0)$ in its surrounding. This changes the electron density $n(x)$ at point x. Let $p(x)$ be the new fermi momentum at x. We have

$$e\phi(x) + p(x)^2/2m = \varepsilon_F = p_F^2/2m \tag{8.21}$$

$$n(x) = 2 \times \frac{4\pi}{3}\frac{p^3}{(2\pi)^3} \tag{8.22}$$

while the background positive ion density n_0 is not affected,

$$n_0 = \frac{8\pi}{3}\frac{p_F^3}{(2\pi)^3}. \tag{8.23}$$

The net charge density is

$$\begin{aligned} e(n(x) - n_0) &= \frac{e}{3\pi^2}(p^3 - p_F^3) \\ &= \frac{e}{3\pi^2}\left[(p_F^2 - 2me\phi)^{3/2} - p_F^3\right] \qquad (8.24) \\ &\approx -\frac{me^2 p_F}{\pi^2}\phi. \end{aligned}$$

In (8.24), we only keep terms linear in ϕ. The Poisson equation gives

$$\begin{aligned} -\nabla^2\phi &= e\delta(x) - \frac{me^2 p_F}{\pi^2}\phi \\ &= -\lambda^2\phi + e\delta(x) \end{aligned}$$

or

$$(-\nabla^2 + \lambda^2)\,\phi = e\delta(x). \tag{8.25}$$

Solving (8.25), we obtain

$$\phi(x) = \frac{e}{4\pi r}e^{-\lambda r} \quad , \quad r = |\mathbf{x}|,$$

as desired.

Even though the result is neat, unfortunately, it is not completely correct. The above result works only in the average sense. The sharp cutoff near the fermi surface makes its fourier transform oscillate, as in the case of a light travelling around a sharp edge. This causes the electron density $n(x)$ to oscillate in space — a fact which can become very important under appropriate situations.

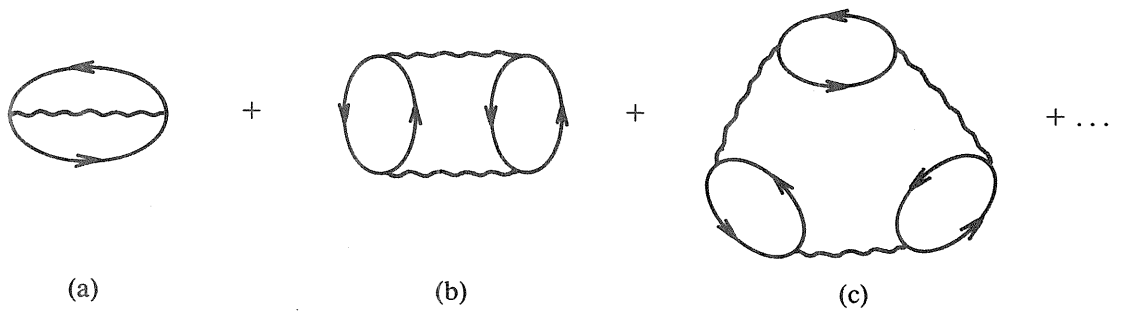


Fig. 8.6 Sum of a selected set of vacuum diagrams.

We mention briefly another place that the long range potential may cause trouble. This is in the ground state energy calculation for the connected diagram shown in Fig. 8.6. We encounter the same $\Pi(q)$, but we also encounter a symmetry factor $1/S$ where $S = 2\times$ (the number of bubbles),

$$-i\varepsilon_a = \frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\, U(q)\,\Pi(q) \tag{8.26a}$$

$$-i\varepsilon_b = \frac{1}{4}\int \frac{d^4q}{(2\pi)^4}\,(U(q)\,\Pi(q))^2 \tag{8.26b}$$

$$-i\varepsilon_c = \frac{1}{6}\int \frac{d^4q}{(2\pi)^4}\,(U(q)\,\Pi(q))^3 \tag{8.26c}$$

$$\ldots \quad .$$

By a power counting at small q, we find that ε_a is finite, ε_b diverges logarithmically, ε_c diverges quadratically, etc. However, the sum of all these contributions is finite for the small q-integration,

$$-i\varepsilon \equiv -i\,(\varepsilon_a + \varepsilon_b + \varepsilon_c + \ldots)$$

$$= -\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\ln\left[1 - U(q)\,\Pi(q)\right]. \tag{8.27}$$

Physically, this is due to the same screening effect as described earlier.

4. Decay of Electrons and Holes

In the electron gas, the electrons and holes are not stable due to the scattering of the electrons in the fermi sea. For instance, an electron can

collide with another electron in the fermi sea and knock it out of the fermi sea, as shown in Fig. 8.7.

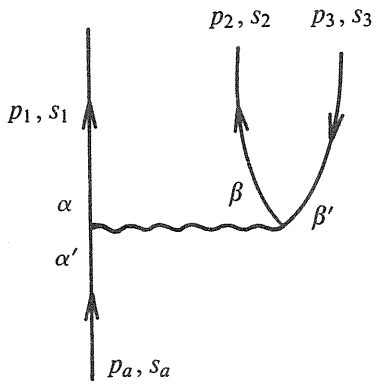


Fig. 8.7 Decay of a particle into a particle and a particle-hole pair.

In Fig. 8.7, particles 1 and 2 are electrons above the fermi sea, and particle 3 is a hole. The invariant amplitude for the process is

$$\mathcal{M} = u^*(s_1)_\alpha \, u(s_a)_{\alpha'} \, U_{\alpha\alpha',\beta\beta'} \, (p_a - p_1) \, u^*(s_2)_\beta \, u(s_3)_{\beta'} \tag{8.28}$$

where $u(s)$, $u^*(s)$ are 2-component spinors with spin s. We need to work out the energy conservation carefully. In the presence of the fermi sea, we wish to measure the energy from the fermi energy ε_F. The incident particle a has an energy $E(p_a) - \varepsilon_F$, and the final particles and hole have energies $E(p_1) - \varepsilon_F$, $E(p_2) - \varepsilon_F$, and $\varepsilon_F - E(p_3)$ respectively where $E(p) = p^2/2m$. The energy conservation implies

$$E(p_a) - \varepsilon_F = E(p_1) - \varepsilon_F + E(p_2) - \varepsilon_F + \varepsilon_F - E(p_3) \tag{8.29a}$$

or

$$E(p_a) + E(p_3) = E(p_1) + E(p_2). \tag{8.29b}$$

This same consideration also applies to the momentum conservation. As we have mentioned earlier, we can interpret the decaying process as the scattering of the electron p_a with the electron p_3 in the fermi sea. From Eq. (8.28), we can write down the differential decay rate as

$$d\Gamma = |\mathcal{M}|^2 \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3}$$

$$\times (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_a)\,\delta(E(p_1) + E(p_2) - E(p_3) - E(p_a))$$

$$\times \theta(p_1 - p_F)\,\theta(p_2 - p_F)\,\theta(p_F - p_3). \tag{8.30}$$

Let us compute the total decay rate near $p_a = p_F$. Note that as $p_a \to p_F$, all p_1, p_2 and $p_3 \to p_F$. Consider

$$U_{\alpha\alpha',\beta\beta'}(q) = \delta_{\alpha\alpha'}\,\delta_{\beta\beta'}\,U(q).$$

Then, we have,

$$\mathcal{M} = u^*(s_1)\,u(s_a)\,u^*(s_2)\,u(s_3)\,U(q). \tag{8.31}$$

The sum over the final state polarizations gives

$$\sum_{s_1 s_2 s_3} |\mathcal{M}|^2 = |U(q)|^2 \sum_{s_1} |u^*(s_1)\,u(s_a)|^2 \sum_{s_2 s_3} |u^*(s_2)\,u(s_3)|^2$$

$$= |U(q)|^2 \cdot 1 \cdot 2 = 2\,|U(q)|^2. \tag{8.32}$$

$$d\Gamma = 2\,|U(p_a - p_1)|^2 \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3}$$

$$\times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_a)$$

$$\times \theta(p_a - p_F)\,\theta(p_1 - p_F)\,\theta(p_2 - p_F)\,\theta(p_F - p_3). \tag{8.33}$$

For $p_a \approx p_F$, we introduce the direction cosines for p_1, p_2, p_3, and p_a as n_1, n_2, n_3 and n_a respectively, giving

$$\mathbf{p}_i = p_F\,\mathbf{n}_i \tag{8.34a}$$

$$\mathbf{p}_a = p_F\,\mathbf{n}_a\,. \tag{8.34b}$$

We also introduce

$$\Delta_1 \equiv E(p_1) - \varepsilon_F > 0 \tag{8.35a}$$

$$\Delta_2 \equiv E(p_2) - \varepsilon_F > 0 \tag{8.35b}$$

$$\Delta_3 \equiv \varepsilon_F - E(p_3) > 0\,. \tag{8.35c}$$

We then have, near $p_a = p_F$,

$$d\Gamma = 2\,|\,U(\mathbf{p}_a - \mathbf{p}_1)\,|^2 \frac{p_1^2\ dp_1\ d\Omega_1}{(2\pi)^3} \frac{p_2^2\ dp_2\ d\Omega_2}{(2\pi)^3}$$

$$\times \frac{p_3^2\ dp_3\ d\Omega_3}{(2\pi)^3} (2\pi)^4 \frac{1}{p_F^3} \delta^3(n_1 + n_2 - n_3 - n_a)$$

$$\times \delta(\Delta_1 + \Delta_2 + \Delta_3 - (E_a - \varepsilon_F)). \tag{8.36}$$

Using

$$E_1 \equiv E(p_1) = p_1^2/2m, \ldots$$

we have

$$p_1\ dp_1 = m\ dE_1 = m\ d\Delta_1, \text{ etc,}$$

and consequently

$$d\Gamma = 2\,|\,U(p_F(\mathbf{n}_a - \mathbf{n}_1))\,|^2 \frac{m^3}{(2\pi)^5} \delta^3\,(n_1 + n_2 - n_3 - n_a)$$
$$\times\ d\Omega_1\ d\Omega_2\ d\Omega_3\ d\Delta_1\ d\Delta_2\ d\Delta_3\ \delta(\Delta_1 + \Delta_2 + \Delta_3 - (E_a - \varepsilon_F)). \tag{8.37}$$

The total decay rate is

$$\frac{1}{\tau} \equiv \Gamma = 2 \int |\,U(p_F(\mathbf{n}_a - \mathbf{n}_1))\,|^2 \frac{m^3}{(2\pi)^5}$$

$$\times\ \delta^3\,(n_1 + n_2 - n_3 - n_a)\ d\Omega_1\ d\Omega_2\ d\Omega_3$$

$$\times \int_0^{E_a-\varepsilon_F} d\Delta_1\ d\Delta_2\ d\Delta_3\ \delta(\Delta_1 + \Delta_2 + \Delta_3 - (E_a - \varepsilon_F)).$$

$$= (E_a - \varepsilon_F)^2 \int d\Omega_1\ d\Omega_2\ d\Omega_3 \left| U(p_F(\mathbf{n}_a - \mathbf{n}_1)^2) \right|^2$$

$$\times \frac{m^3}{(2\pi)^5} \delta^3 (n_1 + n_2 - n_3 - n_a)$$

$$= \text{const}\ (E_a - \varepsilon_F)^2. \tag{8.38}$$

The lifetime of these particles increases as $1/(E_a - \varepsilon_F)^2$ as $p_a \to p_F$. These long-lived single particle excitations are called quasiparticles. Note that the behavior $1/(E_a - p_F)^2$ is independent of the form of $U(q)$. It is also independent of whether the interaction U has any spin-dependence. Using a similar method, we can show that the hole has a similar lifetime near the fermi surface.

As the particles and holes decay, their energies are shared by other particles and holes near the fermi surface. The net effect is to change the kinetic energy of the incident particle into the thermal energy of the fermi system.

We can associate the self-energy diagram given in Fig. 8.8 with our

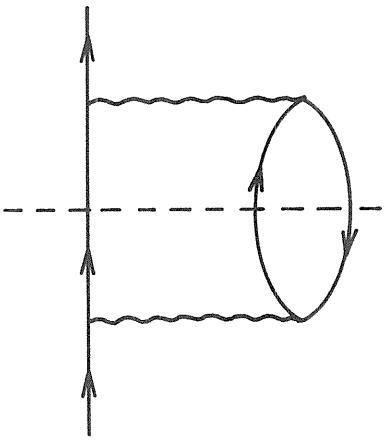


Fig. 8.8 Unitarity diagram associated with the decay process.

decay process. Indeed, the imaginary part of $\Sigma(p)$ in Fig. 8.8 is our decay width Γ. The full propagator generated by Σ in Fig. 8.8 is of the form

$$\frac{i}{p_0 - \varepsilon(p) - i\Gamma}$$

where

$$\varepsilon(p) = \frac{p^2}{2m} + \mathrm{Re}\,\Sigma(p)$$

and

$$\Gamma = \mathrm{Im}\,\Sigma(p).$$

The relation which equates the decay rate (or the cross section) with the imaginary part of an amplitude (here Σ) is known as the unitarity relation. The real part of Σ determines the excitation spectrum.

Chapter 9

BRUEKNER THEORY OF NUCLEAR MATTER

1. Nuclear Matter

The nucleon-nucleon interaction is both strong and short-ranged. To study the nuclear force alone, we can consider a uniform distribution of nuclear medium, and ignore the electromagnetic interactions (the Coulomb interaction) among the nucleons. We refer to this uniform nuclear medium as the nuclear matter. For simplicity, we also suppress the effects of the isospin and the spin. The theory described below was first developed by Bruekner, which also has important application in understanding He.

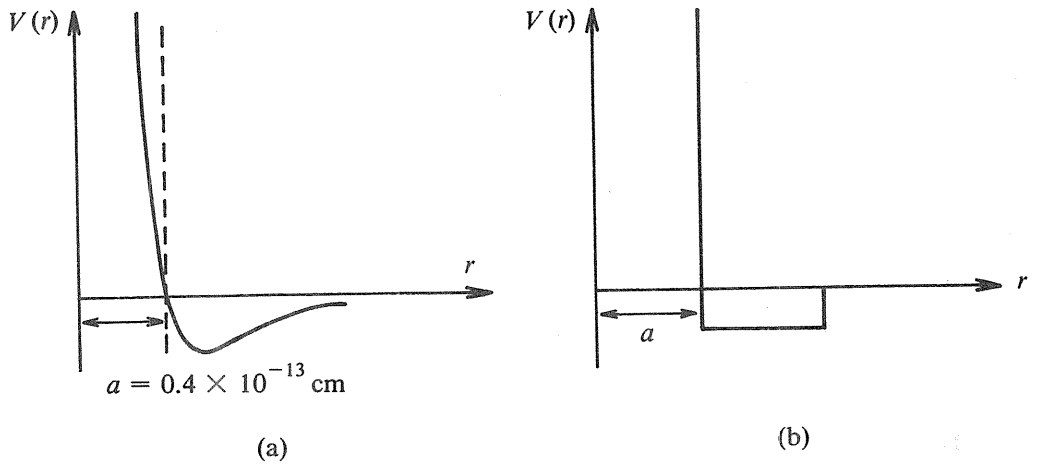


Fig. 9.1 (a) Nucleon-nucleon potential, (b) a simplified potential with hard core.

The nuclear force between two nucleons is strong and repulsive at a short distance, and becomes weakly attractive at large distance. Fig. 9.1 is a simplified version of the nucleon-nucleon potential. As we mentioned earlier, we have ignored the isospin and spin dependence of the potential. To simplify the discussion, we assume that $V(r) = \infty$ for $r < a$. We refer to the infinite repulsive potential as a hard core.

The nucleon density in the nuclear matter associated with the heavy nuclei is relatively low. The average distance between the nucleons is

about 3 times its hard core radius. The hard core occupies approximately 1/27 of the total volume.

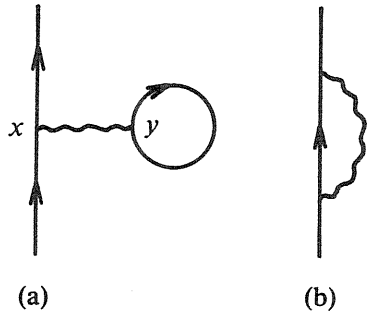


Fig. 9.2 (a) Hartree term, (b) Fock term.

It is easy to see that the Hartree approximation leads to a divergent result. Recall that the Hartree self-energy contribution in Fig. 9.2a is

$$\Sigma_a = \int d^3y\, U(x-y)\, n(y). \tag{9.1}$$

Under the Hartree approximation, $n(y)$ = constant, and the integral diverges as we integrate over the region $|\mathbf{x}-\mathbf{y}| < a$. In the nuclear matter, there is no "background contribution" to cancel this divergence. The screening mechanism does not apply here either because all short-ranged forces are repulsive. We need a different mechanism to avoid this divergence.

2. Bruekner Theory

The divergence arises from the approximation that the nucleon can occupy any space with a uniform probability, independent of the presence of other nucleons. Thus when two nucleons try to occupy the same space, it leads to a divergence. On the other hand, since the potential is infinitely repulsive; there is no possibility for two nucleons to occupy the same space. The Hartree approximation fails because we have not taken the particle-particle correlation into account. Bruekner suggested that we should use the nucleon-nucleon scattering amplitude as an effective potential. The scattering amplitude can be represented graphically in Fig. 9.3.

Bruekner gave an argument why we may ignore the hole contributions as shown in Fig. 9.3. Since the nucleon density in the nucleon matter is low, and since the nucleon force is short-ranged; the contribution from

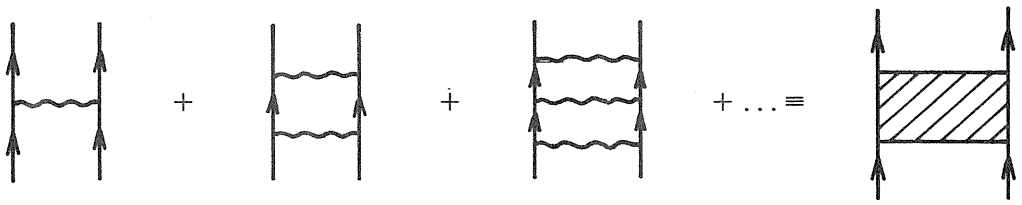


Fig. 9.3 Nucleon-nucleon scattering amplitude as described by the sum of ladder diagrams.

the hole is proportional to $p_F a$ which is small. Thus we need to include as small a number of hole lines as possible. By fixing the hole lines, and summing over the nucleon lines, we obtain automatically the ladder diagrams shown in Fig. 9.3. For example, in the nucleon self-energy diagram, we need to include at least one hole line. In Fig. 9.4, we present two classes of these 1-hole self-energy diagrams. If we can evaluate the ladder amplitudes in Fig. 9.3, we know how to handle the diagrams in Fig. 9.4.

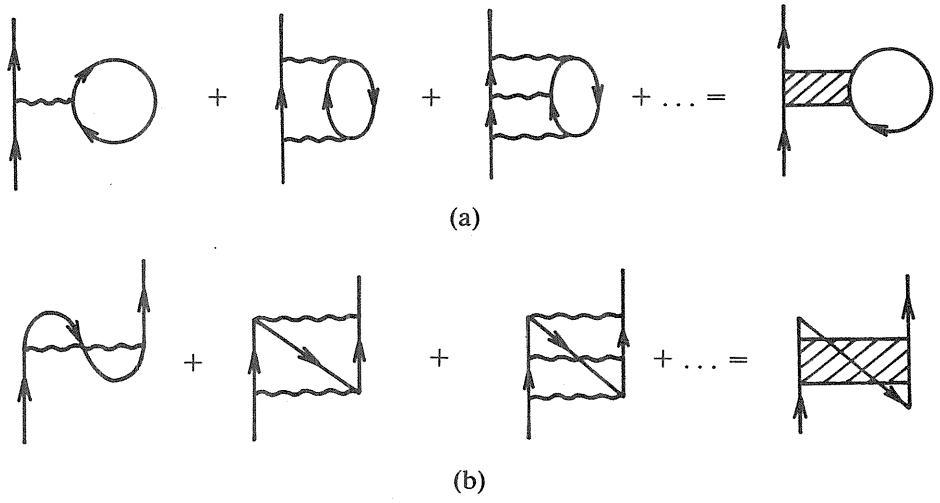
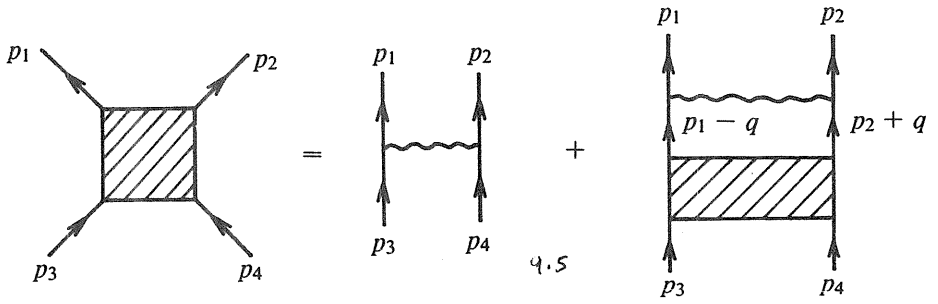


Fig. 9.4 Modified Hartree–Fock terms.

In Fig. 9.5, we establish an integral equation relating the sum of the ladder diagrams Γ to the original potential U. This integral equation is a simple example of the Bethe–Salpeter equation. The integral equation reads,



9.5

Fig. 9.5 Bethe–Salpeter equation for Γ.

$$\Gamma(p_1, p_2; p_3, p_4)$$

$$= U(p_1 - p_3) + (-i)\int \frac{d^4q}{(2\pi)^4} U(q)\, G(p_1 - q)\, G(p_2 + q)\, \Gamma(p_1 - q, p_2 + q; p_3, p_4)$$

$$= U(p_1 - p_3) + i \int \frac{d^4q}{(2\pi)^4} U(q) \frac{1}{p_1^0 - q^0 - E(p_1 - q) + i\varepsilon\, \mathrm{SGN}(|\mathbf{p}_1 - \mathbf{q}_1| - p_F)}$$

$$\times \frac{1}{p_2^0 + q^0 - E(p_2 + q) + i\varepsilon\, \mathrm{SGN}(|\mathbf{p}_2 + \mathbf{q}| - p_F)} \Gamma(p_1 - q, p_2 + q; p_3, p_4). \tag{9.2}$$

In Eq. (8.2), $E(p) = p^2/2M$ is the on-shell energy, and the sign of $i\varepsilon$ $\mathrm{SGN}(p - p_F)$ is determined by whether p is $\gtrless p_F$. For an instantaneous potential, $U(q)$ has no q^0 dependence. We can convince ourselves that $\Gamma(p_1 - q, p_2 + q; p_3, p_4)$ does not depend on q^0. All q^0-dependence is in $G(p_1 - q)$ and $G(p_2 + q)$. The q^0-integration gives

$$\int \frac{dq^0}{2\pi} \frac{i}{p_1^0 - q^0 - E(p_1 - q) + i\varepsilon\, \mathrm{SGN}(|\mathbf{p}_1 - \mathbf{q}| - p_F)}$$

$$\times \frac{1}{p_2^0 - q^0 - E(p_2 + q) + i\varepsilon\, \mathrm{SGN}(|\mathbf{p}_2 + \mathbf{q}| - p_F)}$$

$$= \frac{N}{p_1^0 + p_2^0 - E(p_1 - q) - E(p_2 + q) + i\varepsilon N}$$

$$= \frac{N}{E_0 - E(p_1 - q) - E(p_2 + q) + i\varepsilon N} \tag{9.3}$$

where

$$N = \theta(|\mathbf{p}_1 - \mathbf{q}| - p_F)\, \theta(|\mathbf{p}_2 + \mathbf{q}| - p_F)$$

$$- \theta(p_F - |\mathbf{p}_1 - \mathbf{q}|)\, \theta(p_F - |\mathbf{p}_2 - \mathbf{q}|)$$

$$\approx \theta(|\mathbf{p}_1 - \mathbf{q}| - p_F)\, \theta(|\mathbf{p}_2 + \mathbf{q}| - p_F) \tag{9.4}$$

and

$$E_0 = p_1^0 + p_2^0 = \text{total energy.} \tag{9.5}$$

Hence, we have

$$\Gamma(p_1, p_2; p_3, p_4) = U(p_1 - p_3)$$

$$+ \int \frac{d^3q}{(2\pi)^3} U(q) \frac{N}{E_0 - (E(\mathbf{p}_1 - \mathbf{q}) + E(\mathbf{p}_2 + \mathbf{q})) + i\varepsilon N} \Gamma(p_1 - q, p_2 + q; p_3, p_4) . \tag{9.6}$$

We can write (8.6) compactly as

$$\Gamma = U + U \frac{N}{E_0 - H + i\varepsilon N} \Gamma = U \left(1 + \frac{N}{E_0 - H + i\varepsilon N} \Gamma \right). \tag{9.7}$$

Except for the factor N, Eq. (8.7) is the same as the Lippmann–Schwinger relation,

$$T = U + U \frac{1}{E_0 - H + i\varepsilon} T = U + T \frac{1}{E_0 - H + i\varepsilon} U \tag{9.8}$$

where T is the free space scattering amplitude, and H is the on-shell Hamiltonian. Even through U has a hard core, and becomes infinite for $r < a$, the free space scattering amplitude T is always finite and well-behaved. By eliminating U, we obtain a new integral equation in terms of Γ, and T only.

From (9.8), we obtain

$$T = \left(1 + T \frac{1}{E_0 - H + i\varepsilon} \right) U. \tag{9.9}$$

Multiplying $[1 + T/(E_0 - H + i\varepsilon)]$ to the left of Eq. (9.7), and making use of (9.9), we have

$$\left(1 + T \frac{1}{E_0 - H + i\varepsilon} \right) \Gamma = \left(1 + T \frac{1}{E_0 - H + i\varepsilon} \right) U \left(1 + \frac{N}{E_0 - H + i\varepsilon N} \Gamma \right)$$

$$= T\left(1 + \frac{N}{E_0 - H + i\varepsilon N}\Gamma\right) \tag{9.10}$$

$$= T + T\frac{N}{E_0 - H + i\varepsilon N}\Gamma .$$

After rearranging, we obtain

$$\Gamma = T + T\left(\frac{N}{E_0 - H + i\varepsilon N} - \frac{1}{E_0 - H + i\varepsilon}\right)\Gamma. \tag{9.11}$$

Equation (8.11) is valid even if the underline potential U contains a hard core. This equation is known as Galitsdkii's equation. Since T is finite, we conclude that Γ is always finite.

Under the approximation that the hole contribution is negligible, we have $N \approx 1$, and $\Gamma \approx T$. To the first approximation, we can replace the ladder contribution by the free nucleon-nucleon scattering amplitude T.

Given Γ, we can obtain the ground state energy in the modified Hartree–Fock approximation as in Fig. 9.6, and the self-energy diagrams as in Fig. 9.4.

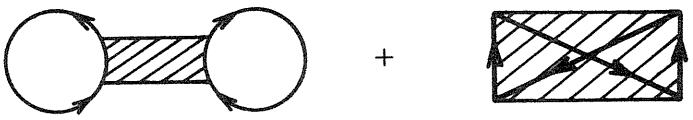


Fig. 9.6 Modified Hartree–Fock ground state diagrams.

As an example, we consider the simplest model with a hard core (Fig. 9.7),

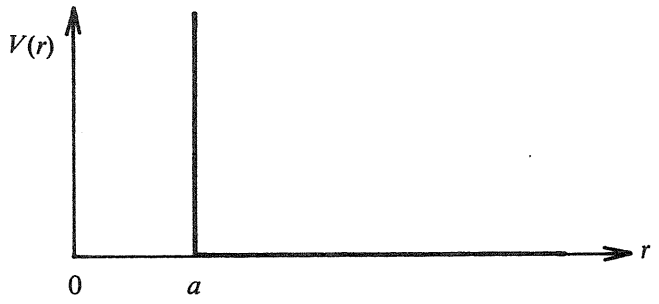


Fig. 9.7 The simplest hard core potential.

The potential is

$$\left.\begin{aligned} V(r) &= 0 \qquad r > a \\ &= \infty \qquad r < a \end{aligned}\right\} \tag{9.12}$$

The scattering amplitude T for this potential is well-known. In the low energy limit, we have

$$T = \frac{4\pi a}{M} - \frac{4\pi i a^2 k}{M} + O(k^2). \tag{9.13}$$

In the low density limit, we obtain

$$\Gamma \approx T = \frac{4\pi a}{M} - \frac{4\pi i a^2 k}{M} + O(p_F k). \tag{9.14}$$

It is straightforward to work out the self-energy diagram in the modified Hartree–Fock approximation,

$$\begin{aligned} \Sigma(p)_{\alpha\alpha'} &= \int_{q<p_F} \frac{d^3q}{(2\pi)^3} [\Gamma_{\alpha\alpha',\beta\beta}(k=0) - \Gamma_{\alpha\beta,\beta\alpha'}(k=p-q)] \\ &\approx \int_{q<p_F} \frac{d^3q}{(2\pi)^3} \left(2 \cdot \frac{4\pi a}{M} - \frac{4\pi a}{M} + \ldots\right) \\ &= \frac{4\pi a}{M} \frac{4\pi}{3} \frac{p_F^3}{(2\pi)^3} + \text{higher order in } (p_F a) \\ &= \frac{2p_F^3 a}{3\pi M} + \text{higher order in } p_F a. \end{aligned} \tag{9.15}$$

This term only shifts the excitation energy by a constant. It has the effect of raising the fermi energy per particle by the same amount.

We can treat the higher order self-energy diagrams in a similar way. We may classify the diagrams by the number of hole lines. This is a regrouping of perturbation series into a different subclass of diagrams. Fig. 9.8 represents a 2-hole contribution to the self-energy diagrams. In Fig. 9.9, we describe the nucleon-hole scattering diagrams with the smallest number of hole lines. After the resummation, we recover all the diagrams. All the results are now finite and, in principle, calculable.

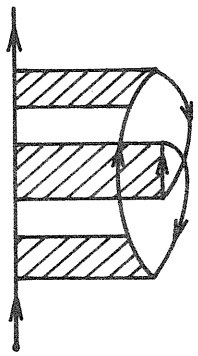


Fig. 9.8 Two-hole contributions to the self-energy diagram.

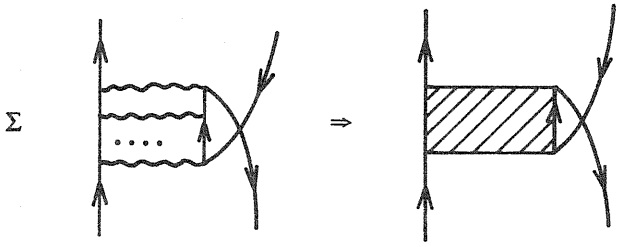


Fig. 9.9 Nucleon-hole scattering amplitude with the smallest number of hole lines.

Chapter 10

ZERO TEMPERATURE BOSE SYSTEM

1. Noninteracting Bose System

Consider an ideal gas of N noninteracting bosons. For an individual particle, we have

$$\varepsilon_k = k^2/2m. \tag{10.1}$$

At the zero temperature, $T = 0$, all bosons condense into a single $k = 0$ state. This phenomenon is known as the bose condensation. During a bose condensation, a finite fraction (here 1) of the particles are in a single state. At a finite but sufficiently low temperature T, the condensation still exists. To find the particle distribution, we introduce a chemical potential μ. Then, the average occupation number of a state is

$$\langle n \rangle = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} \tag{10.2}$$

where $\beta = 1/k_B T$ is the temperature. Only the ground state may have a macroscopic occupation number N_0 with a finite N_0/N. We can replace the summation over the states by

$$\sum_i = \text{contribution due to the } k = 0 \text{ state} + gV \int \frac{d^3k}{(2\pi)^3} \tag{10.3}$$

where $g = 2s + 1$ is the spin multiplicity factor. We have the energy and the particle number as

$$E = 0 + gV \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \varepsilon_k \tag{10.4}$$

$$N = N_0 + gV \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\varepsilon_k - \mu)} - 1} \tag{10.5}$$

where $\varepsilon_k = k^2/2m$ as given in Eq. (10.1).

There are two possible situations.

Case (1): For a sufficiently high T, there is no condensate, and $N_0/N = 0$. We can ignore N_0 in Eq. (10.5) and solve μ as a function of the particle density $n = N/V$,

$$\mu = \mu(n). \tag{10.6}$$

This equation makes sense only if $\mu < 0$.

Case (2): For a sufficiently low temperature, $\mu = 0$. The average occupational number of the lowest energy state diverges, indicating that N_0/N becomes finite.

The condensation starts to occur at $T = T_c$ such that both

$$\frac{N_0}{N} = 0 \qquad \text{and } \mu = 0 \tag{10.7}$$

are satisfied. We can obtain the transition (critical) temperature T in terms of the critical density N/V via

$$N = 0 + gV \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta\varepsilon_k} - 1}$$

$$= gV \int_0^\infty \frac{4\pi k^2\, dk}{(2\pi)^3} \frac{1}{e^{k^2/(2m\, k_B\, T_c)} - 1}. \tag{10.8}$$

Introducing

$$x = k^2/(2m\, k_B\, T_c) \tag{10.9a}$$

$$kdk = mk_B\, T_c\, dx, \tag{10.9b}$$

we have

$$\frac{N}{V} = \frac{g}{2\pi^2} \int_0^\infty \sqrt{2mk_B\, T_c\, x}\; mk_B T_c\, dx \frac{1}{e^x - 1}$$

$$= \frac{g}{4\pi^2} (2mk_B\, T_c)^{3/2} \int_0^\infty \frac{dx\, x^{1/2}}{e^x - 1}$$

$$= \frac{2.315g}{4\pi^2}(2mk_B T_c)^{3/2} \tag{10.10}$$

where the definite integral gives

$$\int_0^\infty \frac{dx\, x^{1/2}}{e^x - 1} = \zeta(3/2)\,\Gamma(3/2) = 2.315.$$

For $T < T_c$, μ remains to be zero, and we have

$$\frac{N - N_0}{V} = g\int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{\varepsilon_k/k_B T} - 1}$$

$$= \frac{2.315g}{4\pi^2}(2mk_B T)^{3/2}. \tag{10.11}$$

Dividing (10.11) by (10.10), we have

$$\frac{N - N_0}{N} = \left(\frac{T}{T_c}\right)^{3/2}$$

or

$$\frac{N_0}{V} = \frac{N}{V}\left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right], \quad (T < T_c). \tag{10.12}$$

Equation (10.12) describes the changes of the condensate density as a function of temperature.

We can work out the energy density of the system for $T < T_c$ as well.

$$\frac{E}{V} = g\int \frac{d^3k}{(2\pi)^3}\frac{\varepsilon_k}{e^{\beta\varepsilon_k} - 1}$$

$$= g\int_0^\infty \frac{4\pi k^2\, dk}{(2\pi)^3}\frac{k^2/2m}{e^{k^2/(2mk_B T)} - 1}$$

$$= \frac{g}{4\pi^2}(2m\,k_B\,T)^{3/2}\,k_B\,T\int_0^\infty \frac{dx\, x^{3/2}}{e^x - 1}$$

$$= \frac{1.783g}{4\pi^2}(2m\,k_B\,T)^{3/2}\,k_B\,T \tag{10.13}$$

where

$$\int_0^\infty \frac{dx\,x^{3/2}}{e^x-1} = \zeta(5/2)\,\Gamma(5/2) = 1.783.$$

The average energy per particle is

$$\frac{E}{N} = \frac{1.783}{2.315}k_B\,T\left(\frac{T}{T_c}\right)^{3/2}$$

$$= 0.770\,k_B\,T\left(\frac{T}{T_c}\right)^{3/2}, \quad (T<T_c). \tag{10.14}$$

Hence, the constant volume heat capacity varies as $T^{3/2}$,

$$C_v = \frac{dE}{dt} = \frac{5}{2}(0.770Nk_B\,(T/T_c)^{3/2}),\ T<T_c\,. \tag{10.15}$$

There are many other interesting properties of the low temperature ideal bose gas.

A simple description of the condensation system is that at $T<T_c$, the system consists of two fluids: The condensate, which is a superfluid, and the rest, which is a normal fluid. The superfluid carries no energy and entropy, and forms a single quantum state. This superfluid has no heat capacity nor viscosity. Helium gives a first approximation of the ideal bose gas.

2. Zero Temperature Interacting Bose System

We consider the interacting system with an interaction Hamiltonian

$$H_{\text{int}} = \frac{1}{2}\int d^3x\,d^3x'\,\psi^\dagger(x)\,\psi^\dagger(x')\,V(x-x')\,\psi(x')\,\psi(x). \tag{10.16}$$

For simplicity, we assume that ψ and $\psi^\dagger$ are scalar fields. The unperturbed ground state is made of the condensate

$$|\phi_0(N)\rangle = |N,0,0,\ldots\rangle \tag{10.17}$$

where all particles are in the lowest energy mode.

The most convenient way to handle the condensate is to consider a ground state with a fixed $\mu = 0$, but with a variable N. This corresponds to changing from a canonical system to a grand canonical system. In such a system, we have

$$\langle \phi_0 | \psi | \phi_0 \rangle = \langle \phi_0 | \psi^\dagger | \phi_0 \rangle = \xi_0 \neq 0 \tag{10.18}$$

where ψ is the field operator for the unperturbed bose system. The expectation of $\psi^\dagger\psi$ becomes nonvanishing, and gives the average particle density n_0 in the condensate,

$$\langle \phi_0 | \psi^\dagger \psi | \phi_0 \rangle = n_0 \approx \xi_0^2 \tag{10.19}$$

or

$$\xi_0 = \sqrt{n_0}\,. \tag{10.20}$$

We can express ψ and $\psi^\dagger$ as

$$\psi(x) \equiv \xi_0 + \phi(x)$$

$$= \xi_0 + \int' \frac{d^3k}{\sqrt{(2\pi)^3}} e^{i\mathbf{k}\cdot\mathbf{x} - ik^2t/2m} a_k \tag{10.21a}$$

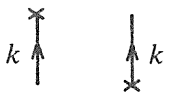
$$\psi^\dagger(x) \equiv \xi_0 + \phi^\dagger(x)$$

$$= \xi_0 + \int' \frac{d^3k}{\sqrt{(2\pi)^3}} e^{-i\mathbf{k}\cdot\mathbf{x} + ik^2t/2m} a_k^\dagger\,. \tag{10.21b}$$

Coefficients a and $a^\dagger$ have the usual meaning of the annihilation and the creation operators. (See Chap. 1, Sec. 4. We have used the continuum normalization here.) The primes on the integrals indicate that we have excluded the $k=0$ mode in the integration.

We can introduce the momentum space Feynman rules as follows.

(a) External particles



amplitude = 1

Fig. 10.1 External particles.

The graphs and their amplitudes are shown in Fig. 10.1. Note that there are no "holes" in a bose system, and no need of introducing antiparticles in a non-relativistic theory. If the particles carry spins, then, we need to include the polarization vectors $\varepsilon(\lambda)$.

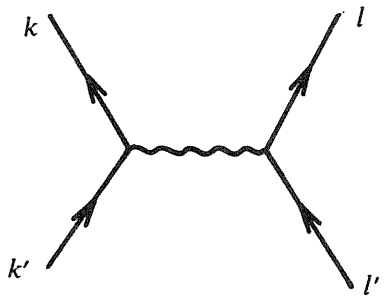
(b) Internal lines

$$\frac{i}{k^0 - \mathbf{k}^2/2m + \mu + i\varepsilon}$$

Fig. 10.2 Propagator.

Since there are no holes or antiparticles, the propagator has a simple i. We include the chemical potential μ for future convenience. For a free bose condensate, $\mu = 0$.

(c) Interaction—No Condensate



$$U(k - k')\,(2\pi)^4\,\delta(k+l-k'-l')$$

Fig. 10.3 Interaction without involving condensate.

If none of the particles has zero momentum, the interaction is given in Fig. 10.3 and has the usual amplitude.

(d) The effect of condensate

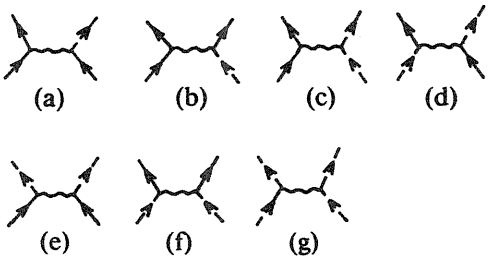


Fig. 10.4 Interaction with particles in the condensate. Dashed lines represent zero-momentum particles in the condensate.

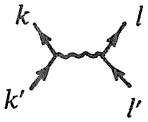
When one or more of the particles have zero momenta, we replace these particles by dashed lines, and assign them the values $\xi_0 = \sqrt{n_0}$. Thus, the process of condensation generates many processes involving zero to three particles (See Fig. 10.4). Note that the one-particle processes do not exist because when three of the particles have zero momenta, the remaining one necessarily has zero momentum too.

(e) Other factors

Fig. 10.5 Hartree contribution.

For an n-interaction process, include a factor $(-i)^n/n!$. When we include all permutations of the interaction potentials, the factor $1/n!$ reduces to $1/S$ where S is the symmetry of the diagram.

Table V. Momentum space Feynman rules for zero-temperature bose system.

Process	Graphical Representation	Feynman amplitude	Remark
External particles		1	$\varepsilon(\lambda)$ if spin $\neq 0$
Internal lines		$\dfrac{i}{k^0-\mathbf{k}^2/2m+\mu+i\varepsilon}$	particles only, no holes.
Interaction no condensate		$U(k-k')$ $\times (2\pi)^4\,\delta(k+l-k'-l')$	
Condensate		$\xi_0 = \sqrt{n_0}$	$k = 0$ term, it does not propagate
n^{th} order		$(-i)^n/n!$ $\int \dfrac{d^4k}{(2\pi)^4}$	Sum over permutation turns $\dfrac{1}{n!} \to \dfrac{1}{S}$, the symmetry factor.

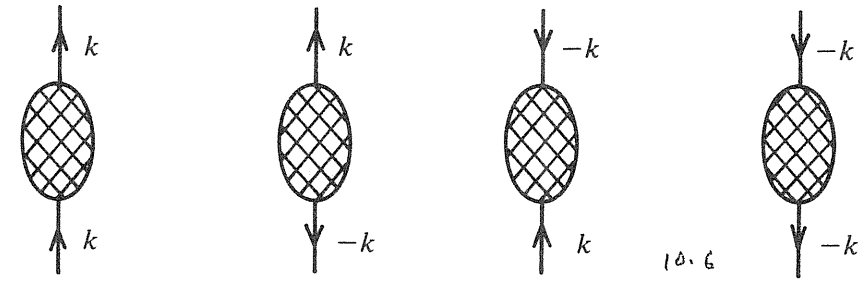
We summarize these Feynman rules in Table V.

Since there are no fermi seas, we do not have the familiar Hartree–Fock diagrams. However, we have terms analogous to the Hartree terms. In Fig. 10.5, we have the pure condensate contribution. The contribution of this term to the ground state energy is

$$\Delta\varepsilon = \frac{1}{2}\xi_0^4\, U(0) = n_0^2\, U(0)/2$$

which is indeed the Hartree term.

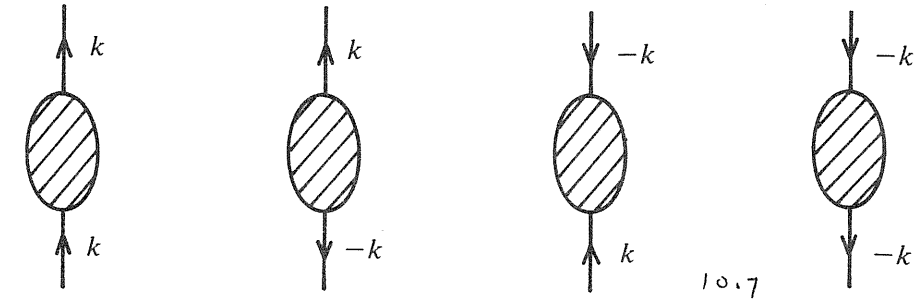
3. Self-Energy Diagrams



$-i\Sigma_{11}(k)$ $-i\Sigma_{12}(k)$ $-i\Sigma_{21}(k)$ $-i\Sigma_{22}(k)$

(a) (b) (c) (d)

Fig. 10.6 Four types of proper self-energy diagrams.



$-iG_{11}(k)$ $-iG_{12}(k)$ $-iG_{21}(k)$ $-iG_{22}(k)$

(a) (b) (c) (d)

Fig. 10.7 Four types of full propagators.

Since the particle number is not conserved, we encounter 4 different types of (proper and improper) self-energy diagrams shown in Fig. 10.6 and Fig. 10.7. In these two diagrams, we label all the momenta along the particle lines. The actual momentum is always pointing up, and is conserved for each of the diagrams. We define a proper self-energy diagram as a self-energy diagram which cannot be separated into 2 self-energy diagrams by cutting a single particle line. We have 4 types of proper self-energy diagrams, and denote them as $-i\Sigma_{11}(k)$, $-i\Sigma_{12}(k)$, $-i\Sigma_{21}(k)$ and $-i\Sigma_{22}(k)$, respectively (See Fig. 10.6).

In Fig. 10.7, we denote the full propagators by $G(p)$. The Dyson equation is a matrix equation,

$$iG(k) = iG_0(k) + iG_0(k)\,(-i\Sigma(k))\,iG(k) \tag{10.22}$$

where

$$iG_0(k)_{11} = \frac{i}{k^0 - \mathbf{k}^2/2m + \mu + i\varepsilon}$$

$$iG_0(k)_{22} = \frac{i}{-k^0 - \mathbf{k}^2/2m + \mu + i\varepsilon} \tag{10.23}$$

$$(G_0)_{12} = (G_0)_{21} = 0.$$

Solving (10.22), we have

$$G(k) = [1 - G_0(k)\,\Sigma(k)]^{-1}\,G_0(k)$$

$$= [G_0(k)^{-1} - \Sigma(k)]^{-1}\,. \tag{10.24}$$

We now work out the simplest self-energy diagrams which are second order in the condensate. These diagrams are shown in Figs. 10.8a, b.

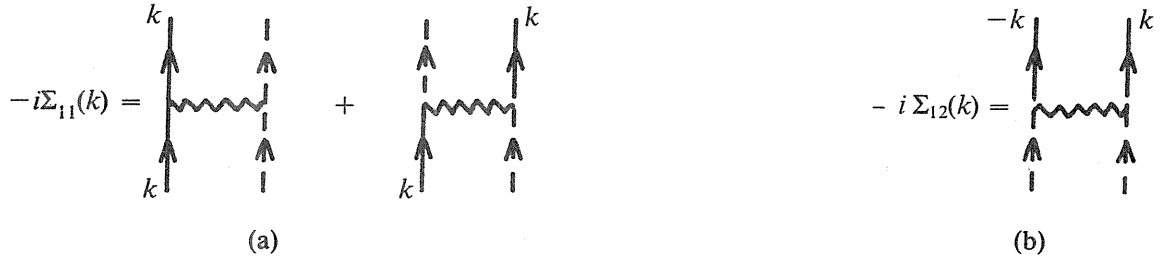


Fig. 10.8 The simplest self-energy diagrams second order in the condensate.

From the Feynman rules, we obtain

$$\Sigma_{11}(k) = n_0\,[V(0) + V(k)] \tag{10.25a}$$

$$\Sigma_{12}(k) = n_0\,V(k)\,. \tag{10.25b}$$

We can work out Σ_{21} and Σ_{22} in a similar way, giving

$$\Sigma_{21}(k) = \Sigma_{12}(-k) = n_0\,V(k) \tag{10.25c}$$

$$\Sigma_{22}(k) = \Sigma_{11}(-k) = n_0 [V(0) + V(k)] . \tag{10.25d}$$

We have assumed that $V(k)$ is a function of $k = |\mathbf{k}|$ only.

The inverse of the full propagator is

$$G^{-1}(k) = G_0^{-1}(k) - \Sigma(k)$$

$$= \begin{bmatrix} k^0 - \mathbf{k}^2/2\mathrm{m} + \mu - n_0 (V(0) + V(k)) & -n_0 V(k) \\ -n_0 V(k) & -k^0 - \mathbf{k}^2/2\mathrm{m} + \mu - n_0 (V(0) + V(k)) \end{bmatrix} . \tag{10.26}$$

It is easy to see that the $n_0 V(0)$ term raises the energy of the lowest 1-particle state by $n_0 V(0)$. Thus, we need to choose $\mu = n_0 V(0)$, giving

$$G^{-1}(k) = \begin{bmatrix} k^0 - \mathbf{k}^2/2\mathrm{m} - n_0 V(k) & -n_0 V(k) \\ -n_0 V(k) & -k^0 - \mathbf{k}^2/2\mathrm{m} - n_0 V(k) \end{bmatrix} . \tag{10.27}$$

We can determine the excitation spectrum by requiring the determinant of $G^{-1}(k) = 0$,

$$\det G^{-1}(k) = -(\mathrm{k}^0)^2 + (k^2/2m + n_0 V(k))^2 - (n_0 V(k))^2 = 0. \tag{10.28}$$

We then obtain the excitation spectrum as

$$\varepsilon(k) \equiv |k^0| = \sqrt{(\mathbf{k}^2/2m)^2 + 2 n_0 V(k) \mathbf{k}^2/2m} . \tag{10.29}$$

It is important to note that, as $k \to 0$, $\varepsilon(k)$ depends linearly on k,

$$\varepsilon(k) = k \sqrt{n_0 V(0)/m} = ck \tag{10.30}$$

with

$$c \equiv \sqrt{n_0 V(0)/m} . \tag{10.31}$$

We can compute its effective mass as

$$m^* = \lim_{k \to 0} k/(\partial \varepsilon(k)/\partial k)$$

$$= \lim_{k \to 0} k/c = 0. \tag{10.32}$$

Thus, in the presence of the bose condensate, the exciton behaves as a massless particle travelling with the sound velocity c (the phonon mode). The linear energy spectrum is extremely important in understanding the superfluidity.

From (10.27) and (10.29), we can determine the eigenstate associated with the excitation spectrum $k^0 = \varepsilon(k)$,

$$G^{-1}(k)\begin{pmatrix} u(k) \\ -v(k) \end{pmatrix} = \begin{pmatrix} \varepsilon(k) - \varepsilon_0(k) - n_0 V(k) & -n_0 V(k) \\ -n_0 V(k) & (\varepsilon k) - \varepsilon_0(k) - n_0 V(k) \end{pmatrix}\begin{pmatrix} u(k) \\ -v(k) \end{pmatrix}$$

$$= 0, \tag{10.33}$$

where $\varepsilon_0(k) = k^2/2m$. We can solve (10.33) easily as

$$u(k) = \sqrt{\frac{\varepsilon_0(k) + n_0 V(k) + \varepsilon(k)}{2\varepsilon(k)}} \tag{10.34a}$$

$$v(k) = \sqrt{\frac{\varepsilon_0(k) + n_0 V(k) - \varepsilon(k)}{2\varepsilon(k)}} \tag{10.34b}$$

and u and v obey the normalization condition

$$u(k)^2 - v(k)^2 = 1. \tag{10.35}$$

We include the negative sign, $-v$, in (10.33) in order to obtain a positive v in Eq. (10.34b).

Next, we compute the annihilation and creation operators associated with the eigenstates. They are

$$\alpha(k) = u(k)\, a(k) - v(k)\, a^\dagger(-k) \tag{10.36a}$$

$$\alpha^\dagger(k) = u(k)\, a^\dagger(k) - v(k)\, a(-k). \tag{10.36b}$$

It is easy to see that

$$[\alpha(k), \alpha(k')] = [\alpha^\dagger(k), \alpha^\dagger(k')] = 0 \tag{10.37a}$$

and

$$[\alpha(k), \alpha^\dagger(k')] = u(k)\, u(k')\, [a(k), a^\dagger(k')]$$

$$+ v(k)\, v(k')\, [a^\dagger(-k), a(-k')]$$

$$= (u(k)^2 - v(k)^2)\, \delta^3(k-k')$$

$$= \delta^3(k-k'). \qquad (10.37b)$$

The normalization condition (10.35) is crucial for α, $\alpha^\dagger$ to have the proper interpretations of the annihilation and creation operators. Equation (10.36) is known as the Bogoliubov transformation. This transformation diagonalizes the quadratic part of the full Hamiltonian in the presence of the condensate. One can verify this by a direct calculation.

Chapter 11

LIQUID HELIUM-4

1. Review of Facts

Helium is the only element which remains liquid at zero temperature. We shall concentrate on He^4 which is a bose system. The He^4 has the phase diagram shown in Fig. 11.1.

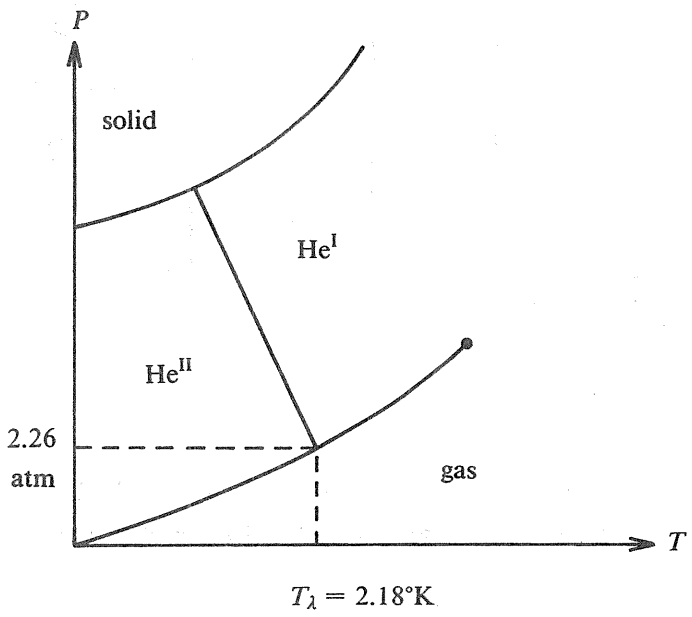


Fig. 11.1 Phase diagrams for He (not in scale).

Helium–I is a normal fluid, and has a normal gas-liquid critical point. Helium–II is a mixture of normal fluid and superfluid. The latter is a bose condensate. Helium-I and Helium-II are separated by a λ-transition line. Helium-I, Helium-II, and vapor coexist at the λ-point,

$$T_\lambda = 2.18°\text{K}.$$

Experimentally, the specific heat of liquid He^4 along the vapor pressure curve forms a logarithmic discontinuity. This logarithmic behavior reflects the onset of bose condensation. (See Fig. 11.2)

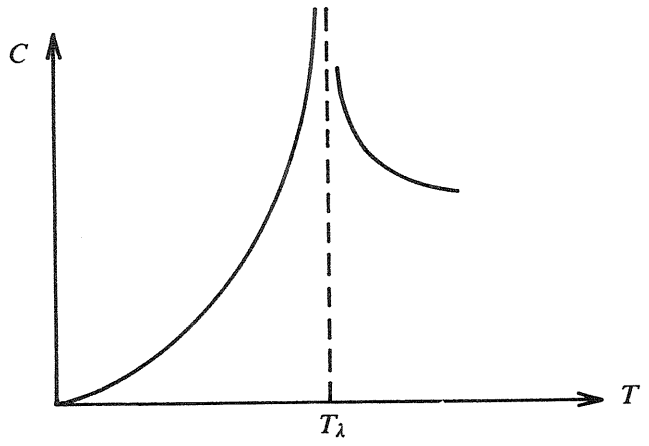


Fig. 11.2 Specific heat near the λ-transition.

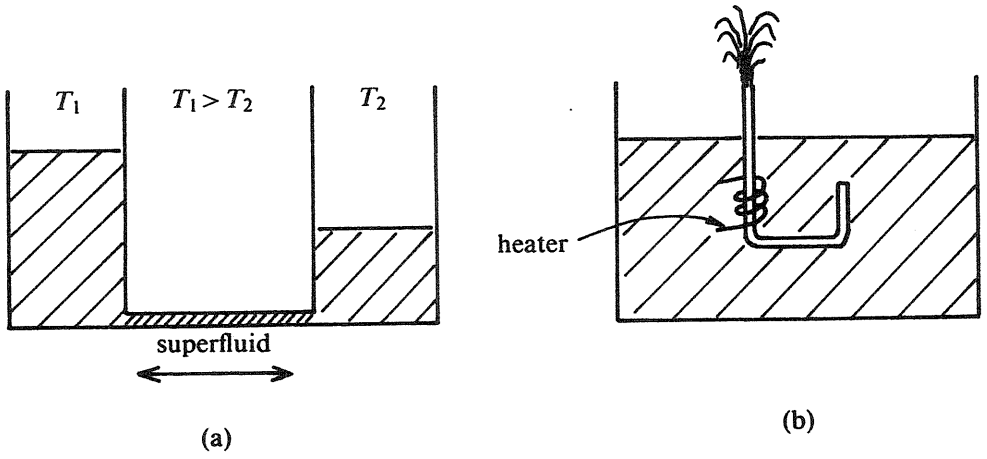


Fig. 11.3 (a) Increasing the temperature of one tank causes the superfluid of the other tank to flow toward it through a connecting capillary. (b) Fountain effect.

Another important fact of He^4 below T_c is that the superfluid part of He^4 has zero viscosity. Consider two containers linked by a capillary through which only the superfluid can pass freely (Fig. 11.3a). Raising the temperature of one tank-converts part of its superfluid to normal fluid. This causes the superfluid of the other tank to move toward it. Indeed, if we heat up one side of the small U-shape tube as shown in Fig. 11.3b, the increase of pressure forces the fluid to eject upward, and gives rise to the fountain effect.

In Chap. 10, we have shown that the excitation energy spectrum $\varepsilon(p)$ for a bose condensate is linear at small p. As we increase the momentum p, both the theoretical and the experimental work give a spectrum as shown in Fig. 11.4.

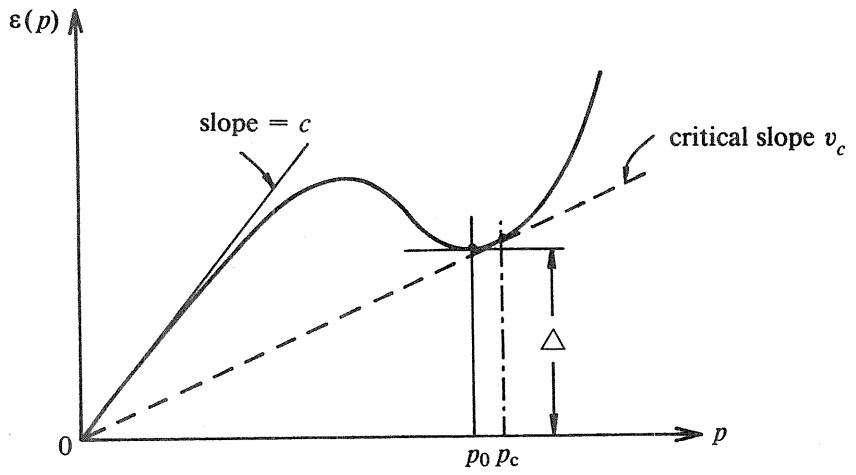


Fig. 11.4 Excitation spectrum of He in the phonon and roton region.

For small p and for p near the dip p_0, we have

$$\varepsilon(p) = cp, \qquad p << p_0 \tag{11.1}$$

$$\varepsilon(p) = \Delta + (p-p_0)^2/2\mu, \qquad p \sim p_0 \tag{11.2}$$

respectively. We refer the excitation for small p as the phonon region, and that near $p = p_0$ as the roton region. The experimental parameters for the phonon and the roton regions are:

$$c = 226 \text{ m/s}$$

$$p_0 = 2 \text{ Å}^{-1}$$

$$\Delta/k_B = 9°\text{K}$$

$$\mu = 0.25\, m_{He}\,.$$

Parameter c represents the sound velocity. There is another important velocity parameter, known as the critical velocity v_c. It is the minimal slope in Fig. 11.4 that a line passing through the origin can reach the roton spectrum curve. Experimentally, we have $v_c = 60$ m/s. At low temperature, the roton excitations are damped by the Boltzmann factor $\exp(-\Delta/k_B T)$.

According to Feynman, the roton excitation is associated with the collective motion as appearing in a smoke ring, shown in Fig. 11.5. As we move an object in a superfluid at small velocity, there is no viscosity. However, when the velocity reaches the critical velocity v_c, viscosity appears due to the creation of rotons, or equivalently, the smoke rings.

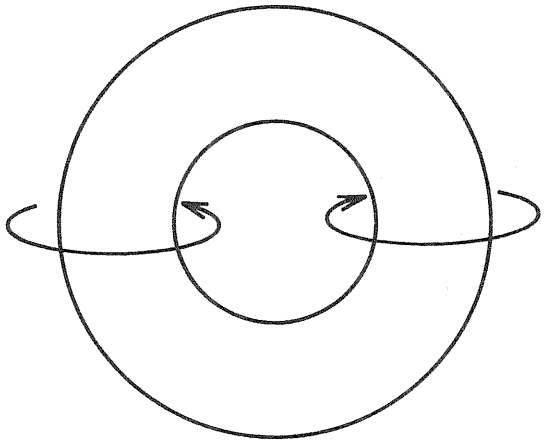


Fig. 11.5 A smoke-ring collective motion.

2. Superfluidity

A superfluid is characterized by the absence of viscosity. We shall first establish that if the excitation spectrum is purely linear, then, the fluid exhibits no viscosity for an object moving with velocity $v < c$. The possible creation of rotons lowers the critical velocity from c to v_c.

Consider a classical object with mass M moving in a superfluid with velocity V. The momentum and energy of this moving object are

$$P = MV \tag{11.3}$$

$$E = p^2/2M. \tag{11.4}$$

We assume that there are no phonons initially. The only way that this object can lose energy is through the emission of one or more phonons. We begin with the emission of a single phonon as described in Fig. 11.6.

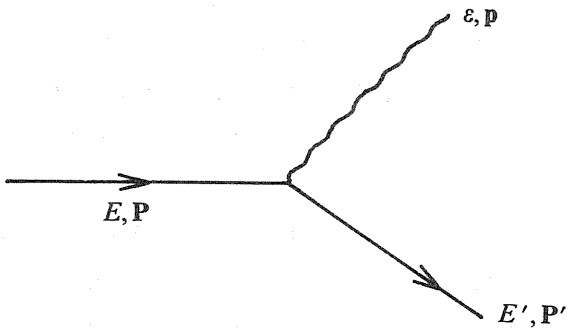


Fig. 11.6 The emission of a single phonon by a moving object. This process is forbidden for $v < c$.

We need to impose some kinematical constraints on this single-phonon emission process. The energy and the momentum conservation imply

$$P^2/2M = P'^2/2\mathrm{M} + cp \tag{11.5}$$

$$\mathbf{P} = \mathbf{P}' + \mathbf{p}. \tag{11.6}$$

Eliminating $\mathbf{P}'$ in (11.5), we have

$$\frac{P^2}{2M} = \frac{(\mathbf{P} - \mathbf{p})^2}{2M} + cp$$

$$= \frac{P^2}{2M} - \mathbf{V}\cdot\mathbf{p} + \frac{p^2}{2M} + cp \tag{11.7}$$

or

$$\mathbf{V}\cdot\mathbf{p} = cp + p^2/2M \geq cp. \tag{11.8}$$

Equation (11.8) indicates that the above process is forbidden when $|\mathbf{V}| < c$. It is easy to see that the same conclusion is valid even if several phonons are emitted, as shown in Fig. 11.6. In the presence of multiphonon emissions, we need to modify Eq. (11.8) to

$$\mathbf{V}\cdot\Sigma\mathbf{p}_i = c\Sigma p_i + (\Sigma\mathbf{p}_i)^2/2M$$

$$\geq c\Sigma p_i\,. \tag{11.9}$$

Inequality (11.9) prevents the process in Fig. 11.7 from happening at $|\mathbf{V}| < c$.

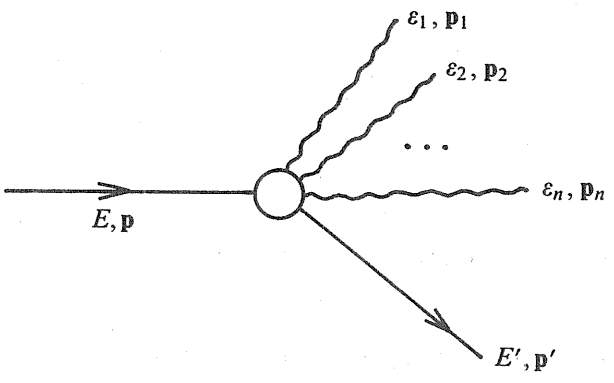


Fig. 11.7 The emission of many phonons by a moving object. This process is also forbidden for $v < c$.

It is strightforward to take the roton and other contributions into account. We only need to replace the phonon spectrum $\varepsilon = cp$ by the measured excitation spectrum, $\varepsilon = \varepsilon(p)$. The inequality (11.8) changes to

$$\mathbf{V}\cdot\mathbf{p} = \varepsilon(p) + p^2/2M \geq \varepsilon(p)$$

or

$$V \geq \varepsilon(p)/p. \tag{11.10}$$

We can achieve the equality in (11.10) by having a heavy object moving with V and by emitting a roton with v parallel to V. The critical velocity v_c is the minimal slope, $\varepsilon(p)/p$, as shown in Fig. 11.4. At V slightly larger than v_c, the moving object begins to lose energy by emitting rotons of momentum p_c and energy $\varepsilon(p_c)$.

A linear spectrum at small p is crucial for the existence of superfluid. If the excitation energy is quadratic, $\varepsilon(p) = p^2/2m$, with $m \neq 0$, then, the viscosity appears as soon as

$$\mathbf{V}\cdot\mathbf{p} \geq p^2/2m$$

or

$$V \geq p/2m.$$

Thus, v_c goes to zero as $p \to 0$ and we no longer have any superfluidity.

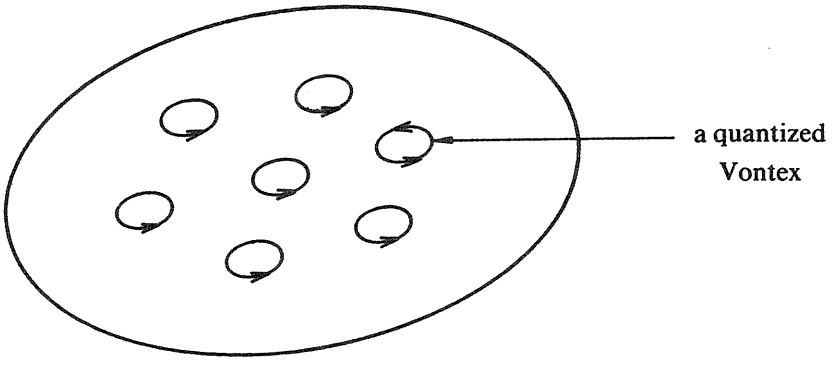


Fig. 11.8 A superfluid with a number of quantized vortices.

A fluid with no viscosity is also rotationless, $\nabla \times \mathbf{v}=0$. It can not support any angular momentum. If we begin with a rotating fluid at $T > T_c$, and then cool it down to $T < T_c$, we shall have a fluid with both $T < T_c$

and a nonvanishing angular momentum. Since the fluid becomes a superfluid at $T<T_c$, where does the angular momentum go? At temperature $T<T_c$, the superfluid state has a lower free energy than the normal fluid state. Nature chooses a clever way of minimizing the free energy and at the same time conserving the angular momentum: It forces the angular momentum into a discrete number of vertex tubes made of normal fluid (See Fig. 11.8). The remaining fluid becomes superfluid. Each of the vortices has a quantized amount of angular momentum, $n\hbar$.

Chapter 12

CLASSICAL SYSTEMS WITH NOISES

The diagrammatical technique is actually very general, and is applicable to some classical systems (non-quantum) as well. We shall first demonstrate that a classical system with noise can mimic a quantum system. We can then derive the Feynman rules from the equivalent quantum system.

1. Equivalent Systems

We consider the following two systems.

(1) A classical system with noise

We choose a simple classical system driven by a gaussian noise $f(t)$,

$$\dot{a} = -\mu a + f(t) \tag{12.1}$$

with

$$\langle f(t) f(t') \rangle = F \delta(t-t'). \tag{12.2}$$

The brackets $\langle \ldots \rangle$ indicate statistical averages. We shall describe the meaning of gaussian noise later.

(2) A quantum system

The corresponding quantum system is

$$\dot{a} = -\mu a + Fb \tag{12.3}$$

$$\dot{b} = \mu b \tag{12.4}$$

where a and b obey the commutation relations of an annihilation and a creation operator,

$$[a(t), b(t)] = 1. \tag{12.5}$$

We shall demonstrate that these two systems give rise to the same Green's functions. In case (1), the Green's functions are defined as the statistical averages of a's. In case (2), the Green's functions are defined by the vacuum expectation values of the corresponding time-ordered products.

We consider model (1) first. To define a gaussian noise $f(t)$, we introduce a test function $\phi(t)$, and require

$$\left\langle e^{i\int dt\,\phi(k) f(t)} \right\rangle = \exp\left[-\frac{1}{2} \int dt\, dt'\, \phi(t)\, \phi(t') \left\langle f(t) f(t') \right\rangle \right]. \tag{12.6}$$

If Eq. (12.6) is satisfied for all test function $\phi(t)$, we call $f(t)$ a gaussian noise. Gaussian noises appear naturally in describing the effect of thermal fluctuations.

Expanding Eq. (12.6) and equating coefficients of $\phi(t)$, we obtain

$$\left\langle f \cdots f \right\rangle = 0 \qquad \text{for an odd number of } f\text{'s}, \tag{12.7}$$

$$\begin{aligned} \left\langle f(t_1) f(t_2) f(t_3) f(t_4) \right\rangle &= \left\langle f(t_1) f(t_2) \right\rangle \left\langle f(t_3) f(t_4) \right\rangle \\ &\quad + \left\langle f(t_1) f(t_4) \right\rangle \left\langle f(t_2) f(t_3) \right\rangle \\ &\quad + \left\langle f(t_1) f(t_3) \right\rangle \left\langle f(t_2) f(t_4) \right\rangle, \text{ etc.} \end{aligned} \tag{12.8}$$

Equation (12.8) reveals that the expectation value of 4 f's is the same as that given by the Wick theorem. The Wick expansions also apply to the expectation value of an arbitrary number of f's.

We now compute $\left\langle a(t_1)\, a(t_2) \right\rangle$. We integrate (12.1) by means of an integration factor $\exp(\mu t)$, obtaining

$$\begin{aligned} \frac{d}{dt}\left(e^{\mu t}\, a(t)\right) &= e^{\mu t}\left(\dot{a} + \mu a\right) \\ &= e^{\mu t} f(t). \end{aligned} \tag{12.9}$$

Hence, we have

$$a(t)\, e^{\mu t} = \text{const} + \int_{t_0}^{t} dt'\, e^{\mu t'} f(t'). \tag{12.10}$$

The integration constant is $a_0 \exp(\mu t)$. Dividing out $\exp(\mu t)$, we obtain

$$a(t) = a_0\, e^{-\mu(t-t_0)} + e^{-\mu t} \int_{t_0}^{t} dt'\, e^{\mu t'} f(t'). \tag{12.11}$$

Taking $t_0 \to -\infty$, the dependence on the initial a_0 disappears,

$$a(t) = e^{-\mu t} \int_{-\infty}^{t} dt'\, e^{\mu t'} f(t'). \tag{12.12}$$

From (12.12), we can work out the 2-point Green's function as

$$\begin{aligned} \langle a(t_1)\, a(t_2) \rangle &= e^{-\mu t_1}\, e^{-\mu t_2} \\ &\quad \times \int_{-\infty}^{t_1} e^{\mu t'_1}\, dt'_1 \int_{-\infty}^{t_2} dt'_2\, e^{\mu t'_2} \langle f(t'_1) f(t'_2) \rangle \\ &= F\, e^{-\mu(t_1+t_2)} \int_{-\infty}^{t_1} dt'_1\, e^{\mu t'_1} \\ &\quad \times \int_{-\infty}^{t_2} dt'_2\, e^{\mu t'_2}\, \delta(t'_1 - t'_2)\,. \end{aligned} \tag{12.13}$$

Considering the time sequence $t_1 > t_2$, we have

$$\begin{aligned} \langle a(t_1)\, a(t_2) \rangle &= F\, e^{-\mu(t_1 + t_2)} \int_{-\infty}^{t_2} dt'_1\, e^{2\mu t'_1} \\ &= F\, e^{-\mu(t_1 + t_2)}\, e^{2\mu t_2}/2\mu \\ &= \frac{F}{2\mu}\, e^{-\mu(t_1 - t_2)}\,. \end{aligned} \tag{12.14a}$$

Similary, we have for $t_2 > t_1$,

$$\langle a(t_1)\, a(t_2) \rangle = \frac{F}{2\mu}\, e^{-\mu(t_2 - t_1)}\,. \tag{12.14b}$$

Combining these two cases, we obtain the two-point function as

$$\langle a(t_1)\, a(t_2) \rangle = \frac{F}{2\mu}\, e^{-\mu|t_1 - t_2|}\,. \tag{12.15}$$

Using the defintion of gaussian noise in Eq. (6), we can obtain the generating functional of the Green's functions as

$$\left\langle e^{i\int dt\, j(t)\, a(t)} \right\rangle$$

$$= \exp\left\{-\frac{1}{2}\int dt_1 \int dt_2\, j(t_1)\, j(t_2) \langle a(t_1)\, a(t_2)\rangle\right\}$$

$$= \exp\left[-\frac{F}{4\mu}\int dt_1 \int dt_2\, j(t_1)\, j(t_2)\, e^{-\mu|t_1-t_2|}\right]. \tag{12.16}$$

Equation (12.16) contains the information for all the Green's functions.

Next, we work out the 2-point function for model (2) (the quantum model). We solve (12.4) to give

$$b = b_0\, e^{\mu(t-t_0)}. \tag{12.17}$$

Substituting (12.17) into (12.3), we have

$$\dot{a} + \mu a = F b_0\, e^{\mu(t-t_0)}.$$

or

$$\frac{d}{dt}(e^{\mu t} a) = F b_0\, e^{\mu(2t-t_0)}. \tag{12.18}$$

Hence, we obtain

$$e^{\mu t}\, a = a_0 e^{\mu t_0} + \frac{F b_0 e^{-\mu t_0}}{2\mu}\left(e^{2\mu t} - e^{2\mu t_0}\right),$$

or

$$a(t) = a_0\, e^{-\mu(t-t_0)}$$

$$+ \frac{F b_0}{2\mu}\, e^{\mu(t-t_0)} - \frac{F b_0}{2\mu}\, e^{-\mu(t-t_0)}. \tag{12.19}$$

When we evaluate the vacuum expectation value, we impose

$$a_0 |\,\rangle = 0, \qquad \langle\, | b_0 = 0.$$

We evaluate the two-point function $\langle\, T(a(t_1)\, a(t_2))\,\rangle$ for the time sequence $t_1 > t_2$. Note that the only nonvanishing matrix element for the products of a_0 and b_0 is $\langle\, a_0 b_0\,\rangle = 1$. Then, we have

$$\begin{aligned}\langle\, T(a(t_1)\, a(t_2))\,\rangle &= \langle\, a(t_1)\, a(t_2)\,\rangle \\ &= \frac{F}{2\mu} e^{-\mu(t_1-t_0)}\left[e^{\mu(t_2-t_0)} - e^{-\mu(t_2-t_0)}\right] \\ &= \frac{F}{2\mu}\left[e^{-\mu(t_1-t_2)} - e^{-\mu(t_1-t_0)}\, e^{-\mu(t_2-t_0)}\right]. \end{aligned} \tag{12.20}$$

The second term drops out at $t_0 \to -\infty$, and we have

$$\langle\, T(a(t_1)\, a(t_2))\,\rangle = \frac{F}{2\mu} e^{-\mu|t_1-t_2|} \tag{12.21}$$

as desired. Since $a(t)$ depends linearly on a_0 and b_0, we can show that for any test function $j(t)$, the generating functional for the Green's functions is again given by

$$\begin{aligned}&\langle\, T\left(e^{i\int dt\, j(t)\, a(t)}\right)\rangle_{\text{model 2}} \\ &= \exp\left[-\frac{1}{2}\int dt_1 \int dt_2\, j(t_1)\, j(t_2)\,\langle\, T(a(t_1)\, a(t_2))\,\rangle\right] \\ &= \exp\left[-\frac{F}{4\mu}\int dt_1 \int dt_2\, j(t_1)\, j(t_2)\, e^{-\mu|t_1-t_2|}\right]. \end{aligned} \tag{12.22}$$

Equations (12.16) and (12.22) imply that the Green's functions of these two models are indeed identical.

We can generalize the above result to an arbitrary first order differential equation with noise

$$\dot{a} = \Lambda(a) + f(t) \tag{12.23}$$

where the gaussian noise $f(t)$ obeys

$$\langle f(t) f(t') \rangle = F\delta(t-t'). \tag{12.24}$$

Equation (12.23) is known as the Langevin equation. The above classical system is equivalent to the quantum system,

$$\dot{a} = \Lambda(a) + Fb \tag{12.25a}$$

$$\dot{b} = -b\Lambda'(a) \tag{12.25b}$$

with

$$[a, b] = 1. \tag{12.26}$$

The fact that a quantum system is equivalent to a classical system is very important in quantum field theory calculations. It is very easy to simulate (12.23) on a computer with the help of a random number generator.

2. Feynman Rules

We consider a simple case of (12.25a) by including both a linear and a quadratic term in $\Lambda(a)$, obtaining

$$\dot{a} = -\mu a + \lambda a^2 + f(t). \tag{12.27}$$

We assume as before that f is a gaussian noise obeying

$$\langle f(t) f(t') \rangle = F\delta(t-t'). \tag{12.28}$$

As mentioned in the previous section, Eqs. (12.27) and (12.28) are equivalent to the quantum system,

$$\dot{a} = -\mu a + \lambda a^2 + Fb, \tag{12.29a}$$

$$\dot{b} = -b\Lambda'(a) = \mu b - 2\lambda ba. \tag{12.29b}$$

We introduce a Hamiltonian

$$H = -\mu ba + \lambda ba^2 + Fb^2/2, \tag{12.30}$$

and obtain the Heisenberg equations of motion

$$\dot{a} = [a, H] = -\mu a + \lambda a^2 + Fb \tag{12.31a}$$

$$\dot{b} = [b, H] = \mu b - 2\lambda ba, \tag{12.31b}$$

as required. Note that H in (12.30) is nonhermitian, thus H can give an imaginary energy and an exp $(-\mu t)$ t-dependence.

To introduce the Feynman rules through Dyson–Wick expansion, we define as before

$$H = H_0 + H_{\text{int}}, \tag{12.32}$$

with

$$H_0 = -\mu ba + Fb^2/2 \tag{12.33a}$$

$$H_{\text{int}} = \lambda ba^2 . \tag{12.33b}$$

We introduce the interaction picture via

$$O_I(t) \equiv e^{-tH_0} O(0) e^{tH_0}, \tag{12.34}$$

$$|t\rangle_I = e^{-tH_0} e^{tH} |\rangle, \tag{12.35a}$$

$${}_I\langle t| = \langle| e^{-tH} e^{tH_0} . \tag{12.35b}$$

We then introduce an evolution operator

$$|t_1\rangle_I = U(t_1, t_2)\,|t_2\rangle_I \tag{12.36}$$

with

$$U(t_1, t_2) = T\left(\exp \int_{t_2}^{t_1} dt\, H_I(t)\right). \tag{12.37}$$

In the interaction picture, the operators a and b obey the "free equations"

$$\dot{a} = -\mu a + Fb \tag{12.38a}$$

$$\dot{b} = \mu b \tag{12.38b}$$

which give rise to, as before,

$$a(t) = \left(a_0 - \frac{Fb_0}{2\mu}\right) e^{-\mu(t-t_0)} + \frac{Fb_0}{2\lambda} e^{\mu(t-t_0)}, \tag{12.39a}$$

$$b(t) = b_0 e^{\mu(t-t_0)}. \tag{12.39b}$$

The operators a_0 and b_0 annihilate the vacuum on the right and on the left,

$$a_0 |\,\rangle = 0, \qquad \langle\, | b_0 = 0, \tag{12.40}$$

respectively. By taking $t_0 \to -\infty$, we can work out the free propagators as

$$\langle\, T(a(t_1)\, a(t_2))\,\rangle = \frac{F}{2\mu} e^{-\mu|t_1-t_2|} \tag{12.41a}$$

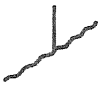
$$\langle\, T(a(t_1)\, b(t_2))\,\rangle = \theta(t_1-t_2)\, e^{-\mu(t_1-t_2)} \tag{12.41b}$$

$$\langle\, T(b(t_1)\, b(t_2))\,\rangle = 0. \tag{12.41c}$$

We may denote the contraction $a(t_1)\, a(t_2)$ by a wiggle line, and the contraction $a(t_1)\, b(t_2)$ by a straight line followed by a wiggle line. We write down the Feynman rules for this simple toy model in Table VI.

Table VI. Feynman rules for the toy model defined by Eqs. (12.32) and (12.33).

Processes	Graphic Representations	Amplitudes
Propagator $a(t_1)\, a(t_2)$		$\frac{F}{2\mu} e^{-\mu\lvert t_1-t_2\rvert}$
Propagator $a(t_1)\, b(t_2)$		$\theta(t_1-t_2)\, e^{-\mu\lvert t_1-t_2\rvert}$

Propagator $\overbrace{b(t_1)\, b(t_2)}$	none	0
Interaction $H_I = \lambda b a^2$		2λ

We can extend the above technique to a realistic system such as that described by the Navier–Stokes equations,

$$\frac{\partial v_i}{\partial t} = \nu \nabla^2 v_i - v_j \frac{\partial}{\partial x_j} v_i - \frac{1}{\rho} \frac{\partial P}{\partial x_i} + f_i , \tag{12.42a}$$

$$\frac{\partial v_i}{\partial x_i} = 0. \tag{12.42b}$$

H. W. Wyld was the first one to derive the Feynman rules for this system. (See H. W. Wyld, *Ann. of Phys.* **14** (1961) 143 and P. C. Martin, E. D. Siggia, and H. A. Rose, *Phys. Rev.* **A8** (1973) 423). Today, turbulence is one of the most important unsolved problems in physics. Since Navier–Stokes equations and other similar partial differential equations are known to give rise to turbulence solutions, Feynman rules may become important tools for studying turbulence.

3. Fokker–Planck Equations and Boltzmann Distribution

In this section, we give a heuristic derivation of the time evolution of the distribution function $\rho(q, t)$ driven by the Langevin equation. In quantum mechanics analog, ρ is $|\psi|^2$.

We begin with Eqs. (12.23) and (12.24)

$$\dot{q} = \Lambda(q) + f(t) \tag{12.43}$$

$$\langle f(t) f(t') \rangle = F\delta(t-t') \tag{12.44}$$

where we have changed symbol a to q. To gain some physical intuition, we may interpret q as a diffusively moving particle under the influence of a random driven force $f(t)$. To derive an equation for ρ, we first consider a pure brownian motion described by $\Lambda(q) = 0$,

$$\dot{q} = f(t). \tag{12.45}$$

We introduct $K(q, t; q', t')$ as the probability distribution for a particle at q at time t if it is known to be at q' at an earlier time t'. As given in the standard Brownian motion, K has a gaussian distribution. In terms of the time difference $\tau \equiv t-t'$, we have

$$K(q, t; q', t') = \sqrt{\frac{1}{2\pi\tau F}} \exp\left[-\frac{1}{2\pi F}(q-q')^2\right] \tag{12.46a}$$

and

$$\int dq\, K(q, t; q', t') = 1. \tag{12.46b}$$

It is easy to see that

$$\frac{\partial}{\partial t} K(q, t; q', t') = \frac{F}{2}\frac{\partial^2}{\partial q^2} K(q, t; q', t'). \tag{12.47}$$

This simply tells us that the Brownian motion leads to a diffusion equation, as expected.

Next, we consider the general case, $\Lambda \neq 0$. We can interpret (12.43) as a particle under a Brownian motion in a medium whose velocity is $\Lambda(q)$. Equation (12.47) is still valid if we replace $\dfrac{\partial}{\partial t}$ by its total derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{\partial}{\partial q}\Lambda(q). \tag{12.48}$$

Then, we obtain

$$\frac{d}{dt} K(q, t; q', t') \equiv \frac{\partial}{\partial t} K(q, t; q', t') + \frac{\partial}{\partial q}[\Lambda(q)\, K(q, t; q', t')]$$

$$= \frac{F}{2}\frac{\partial^2}{\partial q^2} K(q, t; q', t'). \tag{12.49}$$

Since the density distribution at time t is related to its distribution at a previous time t' by

$$\rho(q, t) = \int dq'\, K(q, t; q', t')\, \rho(q', t'), \tag{12.50}$$

and since $K(q, t; q', t')$ obeys a linear partial differential equation, (12.49), we conclude that

$$\frac{\partial \rho(q, t)}{\partial t} = \frac{F}{2} \frac{\partial^2}{\partial q^2} \rho(q, t) - \frac{\partial}{\partial q} [\Lambda(q) \rho(q, t)]. \tag{12.51}$$

Equation (12.51) is known as Fokker–Planck equation. Recall that $\rho = |\psi|^2$. We also need the phase before we can obtain the equation of ψ. However, for most physical applications, we only measure ρ.

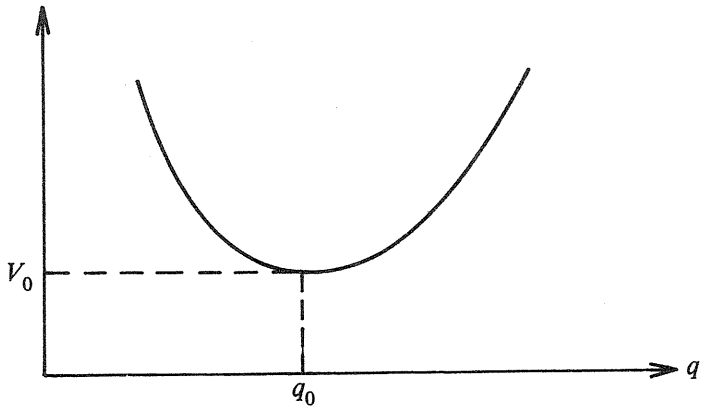


Fig. 12.1 Potential $V(q)$ in the neighborhood of its absolute minimum q_0.

We consider a conservative force which is the gradient of a potential $V(q)$,

$$\Lambda(q) = -\frac{dV(q)}{dq}. \tag{12.52}$$

We consider the steady state distribution near the absolute minimum of $V(q)$ (See Fig. 12.1). For a steady state distribution, we have

$$\frac{\partial \rho(q, t)}{\partial t} = 0 \tag{12.53}$$

and Eq. (12.51) reduces to

$$\frac{\partial}{\partial q} \left[\frac{F}{2} \frac{\partial \rho}{\partial q} - \Lambda(q) \rho \right] = 0. \tag{12.54}$$

We integrate (12.54) to give

$$\frac{F}{2}\frac{\partial \rho}{\partial q} - \Lambda(q)\,\rho(q) = \text{const.} \tag{12.55}$$

Since $\rho = 0$ and $\dfrac{\partial \rho}{\partial q} = 0$ at $q \to \pm\infty$, the integration constant is zero. Hence, we have

$$\frac{1}{\rho}\frac{\partial \rho}{\partial q} = \frac{2}{F}\Lambda(q) = -\frac{2}{F}\frac{dV(q)}{dq}$$

which implies

$$\ln \rho = -\frac{2}{F}V(q) + \text{const}$$

or

$$\rho = \text{const}\, e^{-\beta V(q)} \tag{12.56}$$

with

$$\beta\,(\equiv 1/k_{\mathrm{B}}T) = 2/F. \tag{12.57}$$

The equilibrium $\rho(q)$ obeys the Boltzmann distribution with $F/2$ being the temperature. One can show that all other eigenstates are transit states and decay exponentially in time. For a $V(q)$ having several local minima, there exist other eigenstates which may live for a long time. The eigenvalues of Eq. (12.51) associated with these long-lived states describe the tunnelling rates from these local minima to the true minimum.

We can generalize the method described in this Chapter to systems with many (or infinite) degrees of freedom straightforwardly.

Part II

ADVANCED FIELD THEORY

Chapter 1

FIELD QUANTIZATIONS

The purpose of this chapter is to introduce Schwinger's action principle. We shall use it to quantize the interaction fields. Schwinger action principle emphasizes the differential aspect of the action principle. In Chap. 4, we shall study the Feynman path integral which describes the global aspect of the same principle.

1. Action Principle

We begin with a non-relativistic quantum system described by

$$H = p^2/2m + V(q) \tag{1.1}$$

with the equal-time quantization relation

$$[q, p] = i. \tag{1.2}$$

We choose the unit such that $c = \hbar = 1$.

The Heisenberg's equations of motion are

$$\dot{q} = i\,[H, q] \tag{1.3a}$$

$$\dot{p} = i\,[H, p]. \tag{1.3b}$$

In the present case, the quantization rule is straightforward. It is given by replacing the classical Poisson brackets

$$\{p, q\}_{\text{PB}}$$

with the commutator

$$[p, q]/i.$$

In the case in which the conjugate variables are not known, we need a more general method for imposing the quantization relation.

Consider the Lagrange formulation

$$L = T - V = \frac{1}{2} m\dot{q}^2 - V(q). \tag{1.4}$$

The action is

$$W = \int_{t_1}^{t_2} dt\, L(q, \dot{q}). \tag{1.5}$$

Action principle tells us that W is stationary under small variations,

$$\begin{aligned} \delta W &= \int_{t_1}^{t_2} dt \left[\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right] \\ &= \frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} + \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q \\ &= G(t_2) - G(t_1) \end{aligned} \tag{1.6}$$

where

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \tag{1.7}$$

and

$$G_{\delta q}(t) \equiv \frac{\partial L}{\partial \dot{q}} \delta q = p \delta q. \tag{1.8}$$

An important aspect of Schwinger's action principle is that it also supplies the canonical quantization relations. The surface term $G(t)$ of δW, which is ignored in the conventional action principle, actually provides the quantization relation. Schwinger interpreted these surface terms as the generators for the unitary transformations which induce the variations.

To see the justification of the above identification, we know from the classical mechanics that for an arbitrary function of q,

$$\delta F(q) = \{G_{\delta q}, F\}_{\text{PB.}}\,. \tag{1.9}$$

Schwinger generalizes (1.9) to a quantum mechanics system through

$$\delta F = \frac{1}{i}[F, G_{\delta q}]\,. \tag{1.10}$$

The right hand side of (1.10) describes the effect of an infinitesimal unitary transformation $1 - iG_{\delta q}$ where $G_{\delta q}$ is the infinitesimal generator.

To derive the quantization rule from (1.10), we choose $F(q) = q$, obtaining

$$\delta q = \frac{1}{i}[q, p\delta q] = \frac{1}{i}[q, p]\,\delta q. \tag{1.11}$$

Hence, we have

$$[q, p] = i$$

as desired. Note that we choose here δq as a c-number.

We can generalize the above method and Eq. (1.10) to describe a system of particles

$$L = \sum \frac{m_i}{2}\dot{q}_i^2 - V(q_1, q_2, \cdots, q_n)\,. \tag{1.12}$$

Schwinger's action principle implies

$$G_{\delta q}(t) = \sum_i p_i\,\delta q_i \tag{1.13}$$

with

$$p_i = m_i\,\dot{q}_i\,. \qquad (i \text{ not summed}) \tag{1.14}$$

Choosing $F(q) = q_i$ in (1.10) for a fixed i, we have

$$\delta q_i = \frac{1}{i}[q_i, p_j]\,\delta q_j\,. \qquad (j \text{ summed}) \tag{1.15}$$

Since δq_i are all independent c-numbers, we have

$$[q_i, p_j] = i\delta_{ij}\,. \tag{1.16}$$

In classical mechanics, we have many kinds of generating functions. When we add a total derivative to L,

$$L' = L + \frac{d\Phi}{dt}, \tag{1.17}$$

we have the new action

$$W' = \int_{t_1}^{t_2} dt\, L' = W + \Phi(t_2) - \Phi(t_1)\,. \tag{1.18}$$

The action principle applied to W' gives rise to the same equations of motion, but a modified generating function G',

$$\delta W' = G'(2) - G'(1) \tag{1.19}$$

with

$$G' = G + \delta\Phi. \tag{1.20}$$

Choosing $\Phi = -pq$, we have

$$\begin{aligned} G' &= p\delta q - \delta(pq) \\ &= -q\delta p \equiv G_{\delta p}\,. \end{aligned} \tag{1.21}$$

This procedure of changing $G_{\delta q}$ to $G_{\delta p}$ is an example of Legendre transformations. For any function of p, $F = F(p)$, we obtain

$$\delta F = \frac{1}{t}[F, G_{\delta p}]\,. \tag{1.22}$$

We can also derive the quantization rule from (1.22).

It is sometimes more convenient to introduce a symmetric generating function

$$G_s = \frac{1}{2}(G_{\delta q} + G_{\delta p}) = \frac{1}{2}(p\delta q - q\delta p)\,. \tag{1.23}$$

Then, for any function of p and q, we have

$$\frac{1}{2}\delta F = \frac{1}{i}[F, G_s]. \tag{1.24}$$

Equation (1.24) is always applicable even if we do not know how to identify the p's and q's.

2. Quantization of a Scalar Field

A free scalar field obeys the Klein–Gordon equation

$$(\partial^2 + m^2)\phi = \left(\left(\frac{\partial}{\partial t}\right)^2 - \nabla^2 + m^2\right)\phi = 0 \tag{1.25}$$

and can be obtained from the Lagrange function

$$\mathscr{L} = \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 - m^2\phi^2\right]. \tag{1.26}$$

The action W between 2 spacelike surfaces σ_1 and σ_2 is

$$W = \int_{\sigma_1}^{\sigma_2} d^4x \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 - m^2\phi^2\right]. \tag{1.27}$$

We now apply the action principle to (1.27), giving

$$\delta W = \int_{\sigma_1}^{\sigma_2} d^4x \left[\frac{\partial\phi}{\partial t}\frac{\partial}{\partial t}\delta\phi - \nabla\phi\cdot\nabla\delta\phi - m^2\phi\delta\phi\right]$$

$$= G(\sigma_2) - G(\sigma_1)$$

$$+ \int_{\sigma_1}^{\sigma_2} d^4x \left[-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi - m^2\phi\right]\delta\phi. \tag{1.28}$$

The stationary principle of δW under the variation of internal $\delta\phi$ implies (1.25). We can identify the surface terms as the generator

$$G_{\delta\phi}(\sigma) = \int d^3x \left(\frac{\partial\phi}{\partial t}\delta\phi\right) \tag{1.29}$$

where we have chosen a flat surface t = const. We can impose the

quantization rules by requiring

$$\delta F(\phi) = \frac{1}{i}[F, G_{\delta\phi}] \tag{1.30}$$

for an arbitrary function of ϕ on the surface σ. Equation (1.30) has the same mathematical form as Eq. (1.10). In (1.29) and (1.30), $\delta\phi$'s are chosen to be c-numbers. Choosing $F = \phi(x, t)$, we obtain

$$\delta\phi(x, t) = \frac{1}{i}\left[\phi(x, t), \int d^3x' \dot{\phi}(x', t)\, \delta\phi(x', t)\right]$$

$$= \frac{1}{i}\int d^3x' \,[\phi(x, t), \dot{\phi}(x', t)]\, \delta\phi(x', t). \tag{1.31}$$

Since all $\phi(x, t)$'s at a given time t are independent, we have the equal-time quantization relation

$$[\phi(x, t), \dot{\phi}(x', t)] = i\delta^3(x - x'). \tag{1.32}$$

Obviously, we can identify

$$\pi(x, t) \equiv \dot{\phi}(x, t)$$

as the momentum variable conjugate to $\phi(x, t)$.

Since $G_{\delta\phi}$ is the generating function associated with $\delta\phi$, we expect that $\delta\pi = 0$ under $G_{\delta\phi}$, giving

$$\delta\pi(x, t) = \frac{1}{i}\left[\pi(x, t), \int d^3x' \,\pi(x', t)\, \delta\phi(x', t)\right]$$

$$= \frac{1}{i}\int d^3x' \,[\pi(x, t), \pi(x', t)]\, \delta\phi(x', t)$$

$$= 0. \tag{1.33}$$

Equation (1.33) implies the equal-time commutation relation

$$[\pi(x, t), \pi(x', t)] = 0. \tag{1.34}$$

We cannot obtain the equal-time commutation relation $[\phi, \phi]$ from $G_{\delta\phi}$, even though we know that it is zero. To obtain this last commutation relation, we need to introduce either $G_{\delta\pi}$ or G_s,

$$G_{\delta\pi} = -\int d^3x\, \phi(x, t)\, \delta\pi(x, t) \tag{1.35}$$

$$G_s = \frac{1}{2}\int d^3x\, (\pi\delta\phi - \phi\delta\pi). \tag{1.36}$$

For an arbitrary function of ϕ and π on σ, we have

$$\frac{1}{2}\,\delta F(\phi, \pi) = \frac{1}{2}\,[F, G_s]\,. \tag{1.37}$$

Choosing F to be $\phi(x, t)$ and $\pi(x, t)$ in (1.37), we obtain the complete set of quantization rules immediately.

It is also important to know that the generating function $G_{\delta\phi}$ depends only on the $(\dot{\phi})^2$ term in $\mathscr{L}$ and is independent of any non-derivative interaction.

3. Quantization of a Dirac Field

We can obtain the Dirac equation

$$(i\partial\!\!\!/ - m)\,\psi = 0 \tag{1.38}$$

from the Lagrange function

$$\mathscr{L} = \bar{\psi}\,(i\partial\!\!\!/ - m)\psi \tag{1.39}$$

with

$$\bar{\psi} = \psi^\dagger\gamma^0. \tag{1.40}$$

In (1.39), we treat both ψ and $\psi^\dagger$ as independent fields. It is well-known that Dirac fields are anti-commuting at a spacelike separation. In order to preserve these anticommutation relations, we restrict ourselves to those variations $\delta\psi$ and $\delta\psi^\dagger$ which anticommute among themselves and with ψ and $\psi^\dagger$.

Consider the action

$$W = \int_{\sigma_1}^{\sigma_2} d^4x \mathscr{L} = \int_{\sigma_1}^{\sigma_2} d^4x\, \bar{\psi}\,(i\partial\!\!\!/ - m)\psi\,. \tag{1.41}$$

Its variation with respect to $\delta\psi$ gives

$$\begin{aligned}\delta W &= \int_{\sigma_1}^{\sigma_2} d^4x\, \bar{\psi}(i\partial\!\!\!/ - m)\delta\psi \\ &= \left[\int d^3x\, i\psi^\dagger \delta\psi\right]_{t_1}^{t_2} + \int d^4x\, \bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - m)\delta\psi. \end{aligned} \tag{1.42}$$

In (1.42), $\overleftarrow{\partial}$ denotes a derivative acting on objects to its left. Schwinger action principle and (1.42) imply the field equation

$$\bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - m) = 0 \tag{1.43}$$

and the identification of the generating function as

$$G_{\delta\psi} = \int d^3x\, i\psi^\dagger \delta\psi. \tag{1.44}$$

An independent variation $\delta\psi^\dagger$ leads to

$$(i\partial\!\!\!/ - m)\,\psi = 0 \tag{1.45}$$

and a corresponding generator

$$G_{\delta\psi^\dagger} = -\,i \int d^3x\, \delta\psi^\dagger \psi. \tag{1.46}$$

The quantization relation takes the same form for a Dirac field,

$$\begin{aligned}\delta\psi_\alpha(x, t) &= \frac{1}{i}[\psi_\alpha(x, t), G_{\delta\psi}] \\ &= \frac{1}{i}\left[\psi_\alpha(x, t), \int d^3x'\, i\psi_\beta^\dagger(x', t)\,\delta\psi_\beta(x', t)\right] \\ &= \int d^3x'\,(\psi_\alpha(x, t)\,\psi_\beta^\dagger(x', t)\,\delta\psi_\beta(x', t)\end{aligned}$$

$$
\begin{aligned}
&- \psi_\beta^\dagger(x', t)\, \delta\psi_\beta(x', t)\, \psi_\alpha(x, t)) \\
&= \int d^3x' \,(\psi_\alpha(x, t)\, \psi_\beta^\dagger(x', t)\, \delta\psi_\beta(x', t) \\
&+ \psi_\beta^\dagger(x', t)\, \psi_\alpha(x, t)\, \delta\psi_\beta(x', t)) \\
&= \int d^3x' \,\{\psi_\alpha(x, t), \psi_\beta^\dagger(x', t)\}\, \delta\psi_\beta(x', t)\,.
\end{aligned}
\tag{1.47}
$$

Equation (1.47) implies that, at equal time,

$$
\{\psi_\alpha(x, t), \psi_\beta^\dagger(x', t)\} = \delta^3(x - x')\cdot\delta_{\alpha\beta}. \tag{1.48}
$$

Considering the variation $\delta\psi^\dagger$. we also obtain at equal time,

$$
\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0. \tag{1.49}
$$

As we have mentioned earlier, the generating function depends only on the Kinematical terms, $\frac{1}{2}(\partial_\mu\phi)^2$ and $i\psi\partial\!\!\!/\psi$. Thus, the canonical quantization relations are independent of non-derivative interactions. For instance, an interacting Langrange function,

$$
\begin{aligned}
\mathscr{L} &= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\psi \\
&+ \mathscr{L}_{\text{int}}(\psi, \psi^\dagger, \phi)
\end{aligned}
\tag{1.50}
$$

with a non-derivative $\mathscr{L}_{\text{int}}$, gives rise to the generating function

$$
G = \int d^3x\,(\dot{\phi}\delta\phi + i\psi^\dagger\delta\psi)\,. \tag{1.51}
$$

The quantization condition is still

$$
\delta F = \frac{1}{i}[\,F, G\,]
$$

which implies all the canonical quantization relations.

4. Quantization of a Vector Field

The quantization of a vector field contains a number of interesting features which are common in complicated systems. A free massive vector field obeys

$$(\partial^2 + m^2)\,\phi_\mu = 0 \tag{1.52}$$

$$\partial_\mu \phi_\mu = 0. \tag{1.53}$$

Equation (1.52) indicates that the quanta of the field are free particles with mass m. Equation (1.53) implies that the scalar part of the 4-vector vanishes. Equation (1.53) is a simple example of the auxiliary conditions. These auxiliary conditions are necessary to kill the unwanted components of a covariant field. The implementation of these auxiliary conditions consistently is a major challenge of many field theories in particle physics.

In the present case, we can combine (1.52) and (1.53) into a single equation

$$(g_{\mu\nu}(\partial^2 + m^2) - \partial_\mu \partial_\nu)\,\phi_\nu = 0. \tag{1.54}$$

To show that (1.54) is equivalent to (1.52) and (1.53), we differentiate (1.54) with respect to x_μ, and obtain

$$(\partial^2 + m^2)\,\partial_\nu \phi_\nu - \partial^2 \partial_\nu(\phi_\nu) = m^2 \partial_\nu \phi_\nu = 0$$

which is (1.53). Substituting (1.53) into (1.54), we reproduce (1.52).

We can obtain (1.54) from the Lagrange function

$$\mathscr{L} = -\frac{1}{4}\phi_{\mu\nu}\phi_{\mu\nu} + \frac{m^2}{2}\phi_\mu^2 \tag{1.55}$$

with

$$\phi_{\mu\nu} \equiv \partial_\mu \phi_\nu - \partial_\nu \phi_\mu. \tag{1.56}$$

The action principle gives

$$\delta W = \delta \int_{\sigma_1}^{\sigma_2} d^4x \left(-\frac{1}{4}\phi_{\mu\nu}\phi_{\mu\nu} + \frac{1}{2} m^2 \phi_\mu \phi_\mu \right)$$

$$= \int_{\sigma_1}^{\sigma_2} d^4x\,(-\phi_{\mu\nu}\,\partial_\mu\delta\phi_\nu + m^2\phi_\mu\delta\phi_\mu)$$

$$= G(2) - G(1) + \int d^4x\,(\partial_\mu\phi_{\mu\nu} + m^2\phi_\nu)\delta\phi_\nu \qquad (1.57)$$

with the generating function being

$$G = \int d^3x\,(-\phi_{0\nu}\delta\phi_\nu)$$

$$= \int d^3x\,\phi_{0k}\,\delta\phi_k\,. \qquad (1.58)$$

The field equation is

$$\partial_\mu(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu) + m^2\,\phi_\nu = 0 \qquad (1.59)$$

which reduces trivially to (1.54)

In the generating function (1.58), only ϕ_k and ϕ_{0k} appear as independent variables. It is natural for us to identify

$$\pi_k \equiv \phi_{0k} = \partial_0\phi_k - \partial_k\phi_0 \qquad (1.60)$$

as the momentum variable conjugate to ϕ_k. The equal-time quantization relations are

$$[\phi_k(x),\,\phi_l(y)] = [\pi_k(x),\,\pi_l(y)] = 0 \qquad (1.61)$$

and

$$[\phi_k(x),\,\pi_l(y)] = i\delta_{kl}\,\delta^3(x-y). \qquad (1.62)$$

To see that ϕ_0 and ϕ_{kl} are indeed dependent fields, we use the field equations to obtain

$$\phi_{kl} = \partial_k\phi_l - \partial_l\,\phi_k \qquad (1.63)$$

$$\phi_0 = -\frac{1}{m^2}\,\partial_k\phi_{0k}. \qquad (1.64)$$

Thus, given ϕ_k and ϕ_{0k} at any given time, we can determine ϕ_0 and ϕ_{kl} uniquely. Equations (1.63) and (1.64) are known as constraint equations because they do not involve any time derivatives.

There are some subtleties in deriving Feynman rules by Dyson–Wicks expansion. To see the subtleties, we consider a simple system with a vector field ϕ_μ interacting with an external current j_μ,

$$\mathscr{L} = -\frac{1}{4}\phi_{\mu\nu}\phi_{\mu\nu} + \frac{m^2}{2}\phi_\mu^2 - \phi_\mu j_\mu \tag{1.65}$$

where

$$\phi_{\mu\nu} \equiv \partial_\mu \phi_\nu - \partial_\nu \phi_\mu . \tag{1.66}$$

Since the interaction Lagrange function, $\mathscr{L}_{\text{int}} = -\phi_\mu j_\mu$, is non-derivative, the canonical quantization rules are not affected by the interaction.The momentum conjugated to ϕ_k remains to be $\pi_k \equiv \phi_{0k}$. The field equation derived from (1.65) is,

$$\partial_\mu(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu) + m^2\phi_\nu - j_\nu = 0. \tag{1.67}$$

The constraint equation (1.63) is not affected by j_μ, but Eq. (1.63) is modified to

$$\phi_0 = \frac{1}{m^2}(-\partial_k \phi_{0k} + j_0). \tag{1.68}$$

The Hamiltonian density, expressed as a function of independent field variables, is

$$\begin{aligned}\mathscr{H} &= \frac{\partial \mathscr{L}}{\partial \dot{\phi}_k}\dot{\phi}_k - \mathscr{L} \\ &= \phi_{0k}\dot{\phi}_k - \mathscr{L} \\ &= \phi_{0k}(\dot{\phi}_k - \partial_k\phi_0 + \partial_k\phi_0) \\ &\quad - \left[\frac{1}{2}(\phi_{0k})^2 - \frac{1}{4}\phi_{kl}^2 + \frac{m^2}{2}(\phi_0)^2 - \frac{m^2}{2}\phi_k^2 - \phi_\mu j_\mu\right]\end{aligned}$$

$$= \frac{1}{2}(\phi_{0k})^2 + \frac{1}{4}\phi_{kl}^2 + \frac{m^2}{2}\left[\left(\frac{\partial_k \phi_{0k}}{m^2}\right)^2 + \phi_k^2\right]$$

$$+ \left(-\frac{1}{m^2}\partial_k \phi_{0k} j_0 - \phi_k j_k\right) + \frac{1}{2m^2}(j_0)^2$$

$$+ \text{gradient term}$$

$$= \mathscr{H}_0 + \mathscr{H}_{\text{int}} + \text{gradient term} \tag{1.69}$$

where

$$\mathscr{H}_0 = \frac{1}{2}(\phi_{0k})^2 + \frac{1}{4}\phi_{kl}^2 + \frac{m^2}{2}\left[\left(\frac{\partial_k \phi_{0k}}{m^2}\right)^2 + \phi_k^2\right] \tag{1.70}$$

$$\mathscr{H}_{\text{int}} = -\frac{1}{m^2}\partial_k \phi_{0k} j_0 - \phi_k j_k + \frac{1}{2m^2}(j_0)^2 . \tag{1.71}$$

We may ignore the gradient term because it does not contribute to the action. $\mathscr{H}_0$ is the free vector field Hamiltonian density expressed in terms of the independent variables. For a free vector field, $\partial_k \phi_{0k}/m^2$ is ϕ_0, and we may express $\mathscr{H}_{\text{int}}$ as

$$\mathscr{H}_{\text{int}} = \phi_\mu j_\mu + \frac{1}{2m^2}(j_0)^2$$

$$\neq -\mathscr{L}_{\text{int}} . \tag{1.72}$$

In other words, $\mathscr{H}_{\text{int}}$ is not a scalar density.

There is a second place where the Dyson–Wicks program does not produce the anticipated result. This concerns the definition of the propagator function for the free massive vector field.

We can expand the free vector field as

$$\phi_\mu(x) = \int \frac{d^3k}{\sqrt{2\omega_k(2\pi)^3}} \left(e^{-ik\cdot x}\varepsilon_\mu(k,\lambda)\, a(k,\lambda) + e^{ik\cdot x}\varepsilon_\mu(k,\lambda)\, a^\dagger(k,\lambda)\right) \tag{1.73}$$

where the polarization vectors $\varepsilon_\mu(k,\lambda)$ obey

$$\sum_{\lambda} \varepsilon_{\mu}(k,\lambda)\,\varepsilon_{\nu}(k,\lambda) = -g_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{m^2} \tag{1.74a}$$

$$k_{\mu}\varepsilon_{\mu} = 0. \tag{1.74b}$$

The $a(k,\lambda)$ and $a^{\dagger}(k,\lambda)$ are the annihilation and creation operators. We define as usual the propagation function as

$$\begin{aligned}\overbrace{\phi_{\mu}(x)\,\phi_{\nu}(y)} &\equiv \langle\, 0 \,|\, T(\phi_{\mu}(x)\,\phi_{\nu}(y)) \,|\, 0 \,\rangle \\ &= \theta(x^0 - y^0)\,\langle\, 0 \,|\, \phi_{\mu}(x)\,\phi_{\nu}(y) \,|\, 0 \,\rangle \\ &\quad - \theta(y^0 - x^0)\,\langle\, 0 \,|\, \phi_{\nu}(y)\,\phi_{\mu}(x) \,|\, 0 \,\rangle\,. \end{aligned} \tag{1.75}$$

From (1.73) and (1.74), we can show that

$$\overbrace{\phi_k(x)\,\phi_l(y)} = \left(\delta_{kl} - \frac{\partial_k\partial_l}{m^2}\right) i\Delta_F(x-y) \tag{1.76}$$

$$\overbrace{\phi_0(x)\,\phi_k(y)} = -\frac{\partial_0\partial_k}{m^2}\, i\Delta_F(x-y) \tag{1.77}$$

and

$$\overbrace{\phi_0(x)\,\phi_0(y)} = -\frac{\nabla^2}{m^2}\, i\Delta_F(x-y). \tag{1.78}$$

Note that $i\Delta_F(x-y)$ obeys

$$\begin{aligned}(\partial^2 + m^2)\, i\Delta_F(x-y) &= (\partial^2 + m^2)\int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i\varepsilon}\, e^{-ip\,(x-y)} \\ &= (-i)\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)} = -i\delta^4(x-y). \end{aligned} \tag{1.79}$$

We can rewrite (1.78) as

$$\overbrace{\phi_0(x)\,\phi_0(y)} = -\left(1 + \frac{\partial_0^2}{m^2}\right) i\Delta_F(x-y)$$

$$+\frac{1}{m^2}(\partial_0^2-\nabla^2+m^2)\,i\Delta_F(x-y)$$

$$=-\left(1+\frac{\partial_0^2}{m^2}\right)i\Delta_F(x-y)-\frac{1}{m^2}\delta^4(x-y). \quad (1.80)$$

Thus, we have

$$\overbrace{\phi_\mu(x)\,\phi_\nu(y)} = \left(-g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)i\Delta_F(x-y)$$

$$-\frac{i}{m^2}\delta_{\mu 0}\,\delta_{\nu 0}\,\delta^4(x-y). \quad (1.81)$$

The Wicks contraction term is not covariant either. As it turns out in the present case, the noncovariant term in the propagator and the noncovariant term in $\mathscr{H}_{\text{int}}$ cancel each other to all orders of the perturbation series. We can introduce a covariant contraction through

$$\overbrace{\phi_\mu(x)\,\phi_\nu(y)}^{*} \equiv \langle\, 0 \,|\, T^*(\phi_\mu(x)\,\phi_\nu(y)) \,|\, 0\,\rangle$$

$$=\left(-g_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)i\Delta_F(x-y). \quad (1.82)$$

Then, the Dyson–Wicks formulation reduces to a covariant result when we express the S-matrix in terms of T^*-product and $-\mathscr{L}_{\text{int}}$, obtaining

$$S = T\exp\left(-i\int d^4x\,\mathscr{H}_I\right)$$

$$= T^*\exp\left(i\int d^4x\,\mathscr{L}_I\right). \quad (1.83)$$

Unfortunately, this miracle does not always work. In a more complicated theory such as the nonlinear σ model, the cancellation mentioned above is not complete. There is a covariant piece left-over after an analogous cancellation.

References

I. S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg, *Phys. Rev.* **D3** (1971) 2486; T. D. Lee and C. N. Yang, *Phys. Rev.* **128** (1962) 885.

Chapter 2

LORENTZ TRANSFORMATION PROPERTIES

1. Lorentz Invariance

In classical physics, Lorentz invariance implies that all physical laws transform covariantly under a Lorentz transformation. In quantum mechanics, Lorentz invariance has an additional significance. It is known that any transformation which leaves all quantum mechanical $|\text{matrix element}|^2$ invariant has to be either a unitary operator or an antiunitary operator. Antiunitary operator appears only in the discrete transformation. All proper Lorentz transformations are represented by unitary transformations.

Consider an inhomogeneous Lorentz transformation

$$x'_\mu = a_{\mu\nu} x_\nu + b_\mu . \tag{2.1}$$

We denote the corresponding unitary operator in the quantum mechanics Hilbert space by $U(a, b)$. These unitary operators form a representation of the inhomogeneous Lorentz group. Let

$$x' = a_2 x + b_2 \tag{2.2}$$

and

$$x'' = a_1 x' + b_1 \tag{2.3}$$

be two Lorentz transformations. Their combined transformation is

$$\begin{aligned} x'' &= a_1(a_2 x + b_2) + b_1 \\ &= a_1 a_2 x + (a_1 b_2 + b_1) \\ &\equiv a_3 x + b_3 . \end{aligned} \tag{2.4}$$

$U(a, b)$ forms a representation means that

$$U(a_1, b_1)\, U(a_2, b_2) = U(a_3, b_3) \tag{2.5}$$

with (a_3, b_3) given in (2.4).

In a quantum field theory, we expect that the field equations and the canonical quantization rules describe the whole system. In other words, we should be able to construct the unitary operator $U(a, b)$ out of the available field variables. We call a quantum field theory Lorentz invariant if the following 3 conditions are satisfied: (1) We can construct the unitary operators $U(a, b)$ from the field variables. These unitary operators form a representation of an inhomogeneous Lorentz group as described in Eqs. (2.4) and (2.5). (2) All the field equations transform covariantly under these unitary transformations, and (3) The quantization rules are invariant under these transformations. Thus, a quantum field theory provides a framework for verifying its own Lorentz invariance.

At first sight, the verification of Lorentz invariance as mentioned above appears to be an impossible task. Fortunately, we are aided greatly by the mathematicians. The Lorentz transformations form a Lie group. To construct a Lie group, we only need to know the infinitesimal transformations. For an infinitesimal translation

$$x'_\mu = x_\mu + b_\mu$$

we have

$$U(1, b) = 1 + ib_\mu P_\mu + O(b^2). \tag{2.6}$$

For an infinitesimal homogeneous Lorentz transformation,

$$x'_\mu = x_\mu + \varepsilon_{\mu\nu} x_\nu \qquad (a = 1 + \varepsilon),$$

We have

$$U(1 + \varepsilon, 0) = 1 - \frac{i}{2} \varepsilon_{\mu\nu} M_{\mu\nu} + O(\varepsilon_2). \tag{2.7}$$

We refer to P_μ as the energy-momentum operators, M_{0k} as the boost operators, and M_{kl} as the angular momentum operators. However, we also refer to $M_{\mu\nu}$ simply as the (generalized) angular momentum operators. Thus, P_μ and $M_{\mu\nu}$ are the generators of the translations and of

the homogeneous Lorentz transformations (boosts and rotations).

The theory of Lie group provides us with a set of integrability conditions in the form of Lie algebra. For the inhomogeneous Lorentz group, these conditions are

$$[P_\mu, P_\nu] = 0 \tag{2.8}$$

$$\frac{1}{i}[M_{\mu\nu}, P_\lambda] = g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu \tag{2.9}$$

$$\frac{1}{i}[M_{\mu\nu}, M_{\lambda\rho}] = g_{\nu\lambda} M_{\mu\rho} - g_{\mu\lambda} M_{\nu\rho} + g_{\nu\rho} M_{\lambda\mu} - g_{\mu\rho} M_{\lambda\nu}. \tag{2.10}$$

Knowing P_μ and $M_{\mu\nu}$, we can construct $U(a, b)$. Hence, we have reduced the problem of constructing the unitary operators $U(a, b)$ to that of constructing these 10 generators.

Once we have succeeded in constructing P_μ and $M_{\mu\nu}$, it is usually relatively straightforward to verify the other 2 conditions. In general, the field variable $\phi(x)$ is a multicomponent object. Under a Lorentz transformation, both the coordinates and the field components transform linearly into each other. Consider the Lorentz transformation denoted by

$$x' = ax + b \equiv l(x). \tag{2.11}$$

A covariantly transformed $\phi(x)$ obeys

$$U(a, b)\, \phi(x)\, U(a, b)^{-1} = \Lambda^{-1}(a)\, \phi(ax + b) \tag{2.12}$$

or equivalently,

$$U^{-1}(l)\, \phi(x)\, U(l) = \Lambda(a)\, \phi(l^{-1}x). \tag{2.13}$$

The finite dimensional numerical matrix $\Lambda(a)$ forms a representation of the homogeneous Lorentz group

$$\Lambda(a_1)\, \Lambda(a_2) = \Lambda(a_1 a_2). \tag{2.14}$$

To see that (2.14) indeed transforms correctly, we apply two Lorentz transformations to (2.13) and verify that they have the same effect as a single Lorentz transformation of $l = l_1 l_2$:

$$U^{-1}(l_2)\,[U^{-1}(l_1)\,\phi(x)\,U(l_1)]\,U(l_2)$$

$$= \Lambda(a_1)\,U^{-1}(l_2)\,\phi(l_1^{-1}x)\,U(l_2)$$

$$= \Lambda(a_1)\,\Lambda(a_2)\,\phi(l_2^{-1}l_1^{-1}x)$$

$$= \Lambda(a_1 a_2)\,\phi((l_1 l_2)^{-1}x) \tag{2.15}$$

as desired.

If all field variables transform covariantly, then any Lorentz covariant equations will transform correctly.

The group transformation property implies that we only need to establish (2.13) for infinitesimal transformations. These conditions are

$$[\phi(x), P_\mu] = i\partial_\mu \phi(x) \tag{2.16}$$

$$[\phi(x), M_{\mu\nu}] = [i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}]\,\phi(x). \tag{2.17}$$

We refer $L_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ as the orbital angular momentum, and $S_{\mu\nu}$ as the spin angular momentum of ϕ. $L_{\mu\nu}$ and $S_{\mu\nu}$ obey the same commutation relations as $M_{\mu\nu}$, and they work in different spaces. In (2.17), $S_{\mu\nu}$ is the corresponding generator of $\Lambda(a)$. We have reduced the verification of (2.13) into a set of algebraic equations in (2.16) and (2.17).

The verification of quantization rules under $U(a, b)$ is straightforward because the structure of both the commutator and the anticommutator is preserved under a unitary transformation.

2. Free Scalar and Dirac Fields

We now apply our discussion to some concrete cases. We begin with a free scalar field

$$\mathscr{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{m^2}{2}\phi^2. \tag{2.18}$$

Given $\mathscr{L}$, we can compute the stress tensor and hence the energy and momentum operators of the system. The stress tensor for a scalar field is

$$T_{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - g_{\mu\nu}\mathscr{L}$$

$$= \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\mathscr{L}. \tag{2.19}$$

It is easy to verify that

$$T_{\mu\nu} = T_{\nu\mu} \tag{2.20}$$

$$\partial_\mu T_{\mu\nu} = 0 \tag{2.21}$$

and

$$T_{00} = \mathcal{H} = \frac{1}{2}(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2) > 0. \tag{2.22}$$

We shall discuss a more precise way to introduce the stress tensor later.

From the stress tensor, we can introduce the momentum and the angular momentum operators through

$$P_\mu = \int d^3x\, T_{0\mu} \tag{2.23}$$

$$M_{\mu\nu} = \int d^3x\, (x_\mu T_{0\nu} - x_\nu T_{0\mu}). \tag{2.24}$$

Note that only $T_{0\mu}$ enters into the construction of P_μ and $M_{\mu\nu}$.

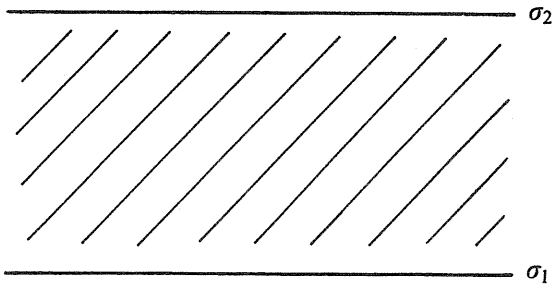


Fig. 2.1 Region bounded by two constant time surfaces σ_1 and σ_2.

Equations (2.20) and (2.21) imply that the energy-momentum operators P_μ and the angular-momentum operator $M_{\mu\nu}$ are both constants of motion. To see that P_μ is conserved, we integrate $\partial_\mu T_{\mu\nu}$ between 2 constant time surfaces, σ_1, σ_2 (Fig. 2.1), and obtain

$$\int_{\sigma_1}^{\sigma_2} d^4x\, \partial_\mu T_{\mu\nu} = 0$$

$$= \int_{\sigma_2} d^3x\, T_{0\nu} - \int_{\sigma_1} d^3x\, T_{0\nu}.$$

Hence, we have

$$\int_{\sigma_1} d^3x\, T_{0\nu} = \int_{\sigma_2} d^3x\, T_{0\nu}. \tag{2.25}$$

To verify that $M_{\mu\nu}$ are constants, we introduce an angular-momentum tensor density

$$m_{\lambda\mu\nu} \equiv x_\mu T_{\lambda\nu} - x_\nu T_{\lambda\mu}. \tag{2.26}$$

It is straightforward to show that

$$\begin{aligned} \partial_\lambda m_{\lambda\mu\nu} &= g_{\mu\lambda} T_{\lambda\nu} - g_{\nu\lambda} T_{\lambda\mu} \\ &= T_{\mu\nu} - T_{\nu\mu} = 0. \end{aligned} \tag{2.27}$$

$M_{\mu\nu}$ is the integral of $m_{0\mu\nu}$.We may interpret $m_{0\mu\nu}$ as the angular-momentum density. Note that to derive (2.27), we need a symmetric and conserved stress tensor.

We have already given the energy density T_{00} in (2.22). The momentum density for a free scalar field is

$$T_{0k} = \dot{\phi}\, \partial_k \phi. \tag{2.28}$$

Hence, we have

$$P_0 = \int d^3x \left[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2\right], \tag{2.29}$$

$$P_k = \int d^3x\, \dot{\phi}\, \partial_k \phi, \tag{2.30}$$

$$M_{0k} = \int d^3x\, (x_0 T_{0k} - x_k T_{00}), \tag{2.31}$$

and

$$M_{kl} = \int d^3x\, \dot{\phi}\, (x_k \partial_l - x_l \partial_k)\phi. \tag{2.32}$$

The operators P_μ and $M_{\mu\nu}$ play two roles in quantum mechanics: (1) Their eigenvalues represent the energy-momentum and the angular

momentum of the sytem, and (2) As operators, they are the generators of the Lorentz transformations.

We can verify that P_μ and $M_{\mu\nu}$ satisfy the algebraic relations (2.8) – (2.10) required as the generators of an inhomogeneous Lorentz group (See home work). In general, the transformation properties under P_k and M_{kl} are easy to verify because they do not involve any time evolution. The transformation under P_0 is the Heisenberg equation of motion. The transformation under M_{0k} is most difficult to verify. This is especially true in the presence of interactions. Thus, the main task of verifying Lorentz invariance is to verify $[M_{0k}, M_{0l}]$ behaves correctly.

We can work out the transformation of a free scalar field $\phi(x)$ under P_μ and $M_{\mu\nu}$ straightforwardly as

$$[\phi(x), P_\mu] = i\partial_\mu \phi(x) \tag{2.33}$$

$$[\phi(x), M_{\mu\nu}] = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\, \phi(x). \tag{2.34}$$

Equations (2.33) and (2.34) are the same as (2.16) and (2.17) with $S_{\mu\nu} = 0$. There are no spin degrees of freedom for a scalar field. We can integrate (2.33) and (2.34) to give

$$U(a, b)^{-1}\, \phi(x)\, U(a, b) = \phi(l^{-1}x) \tag{2.35}$$

where lx is a shorthand notation of

$$(lx)_\mu = a_{\mu\nu}\, x_\nu + b_\mu .$$

Using (2.35) and sandwiching the field equation

$$(\partial^2 + m^2)\, \phi(x) = 0 \tag{2.36}$$

between $U(l)$ and $U(l)^{-1}$, we find that (2.36) transforms covariantly under the Lorentz transformation $x' = lx$. Similary, by sandwiching the canonical quantization relations between $U(l)$ and $U(l)^{-1}$, we verify that the canonical quantizations are invariant under the Lorentz transformation.

We can apply the same technique to a free Dirac field

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{2.37}$$

Here, the stress tensor is

$$T_{\mu\nu} = \frac{1}{2}[\bar{\psi} i \gamma_{(\mu}\partial_{\nu)}\psi - \partial_{(\nu}\bar{\psi} i\gamma_{\mu)}\psi] \tag{2.38}$$

where the parentheses (μ, ν) indicate that the indices μ, ν are symmetrized. It is straightforward to verify that $T_{\mu\nu}$ defined in (2.38) is both symmetric and conserved. From (2.38), we can construct the generators P_μ and $M_{\mu\nu}$. After some complicated algebra, we can verify the Lorentz group commutation relations among P_μ, and $M_{\mu\nu}$. As we have mentioned earlier, the crucial one is to verify the algebra $[M_{0k}, M_{0l}]$.

The transformation properties of $\psi(x)$ under P_μ and $M_{\mu\nu}$ are

$$[\psi, P_\mu] = i\partial_\mu\psi \tag{2.39}$$

$$[\psi, M_{\mu\nu}] = \left[i(x_\mu\phi_\nu - x_\nu\partial_\mu) + \frac{1}{2}\sigma_{\mu\nu}\right]\psi. \tag{2.40}$$

We recognize that $\frac{1}{2}\sigma_{\mu\nu}$ is the intrinsic angular momentum of the Dirac field. From (2.39) and (2.40), we obtain the transformation law of ψ under $U(a, b)$ as

$$U^{-1}(l)\,\psi(x)\,U(l) = \Lambda(a)\,\psi(l^{-1}x) \tag{2.41}$$

with $lx = ax + b$, and

$$\Lambda(a) = \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}\frac{1}{2}\,\sigma_{\mu\nu}\right). \tag{2.42}$$

$\Lambda(a)$ is the finite dimensional (4×4) representation of the homogeneous Lorentz group discussed earlier. A similar transformation for $\psi^\dagger$ is

$$U^{-1}(l)\,\psi^\dagger(x)\,U(l) = \psi^\dagger(l^{-1}x)\,\Lambda^\dagger(a). \tag{2.43}$$

Note that the 4×4 matrix $\Lambda^\dagger$ obeys

$$\Lambda^\dagger = \exp\left[\frac{1}{2}\,i\omega_{\mu\nu}\left(\frac{1}{2}\sigma_{\mu\nu}\right)^\dagger\right]$$

$$= \exp\left[\frac{1}{2}\,i\omega_{\mu\nu}\,\gamma_0\frac{1}{2}\sigma_{\mu\nu}\,\gamma_0\right], \tag{2.44}$$

and

$$\gamma_0 \Lambda^\dagger \gamma_0 = \exp\left[\frac{1}{2} i\omega_{\mu\nu} \frac{1}{2} \sigma_{\mu\nu}\right]$$

$$= \Lambda^{-1}(a). \tag{2.45}$$

Hence, we have

$$U^{-1}(l)\, \bar{\psi}(x)\, U(l) = \bar{\psi}(l^{-1}x)\, \gamma_0 \Lambda^\dagger \gamma_0$$

$$= \bar{\psi}(l^{-1}x)\, \Lambda^{-1}(a). \tag{2.46}$$

Using (2.41) and (2.46), we can work out the Lorentz transformation laws of various bilinear forms such as

$$U^{-1}(l)\, \bar{\psi}(x)\, \gamma_\mu\, \psi(x)\, U(l)$$

$$= U^{-1}(l)\, \bar{\psi}(x)\, U(l)\, \gamma_\mu\, U^{-1}(l)\, \psi(x)\, U(l)$$

$$= \bar{\psi}(l^{-1}x)\, \Lambda^{-1}(a)\, \gamma_\mu\, \Lambda(a)\, \psi(l^{-1}x)$$

$$= a_{\mu\nu}\, \bar{\psi}(l^{-1}x)\, \gamma_\nu\, \psi(l^{-1}x) \tag{2.47}$$

where we have used the 4×4 matrix transformation property

$$\Lambda^{-1}(a)\, \gamma_\mu\, \Lambda(a) = a_{\mu\nu}\, \gamma_\nu . \tag{2.48}$$

Thus, we establish that the bilinear form, $V_\mu(x) \equiv \bar{\psi}(x)\, \gamma_\mu \psi(x)$, transforms as a 4-vector under $U(l)$. We summarize the transformation laws for other bilinear combinations in Table I.

Table I Transformation laws for various bilinear forms.

Bilinear forms		Transformation laws	
$S(x)$	$= \bar{\psi}(x)\, \psi(x)$	$U^{-1} S(x)\, U$	$= S(l^{-1}x)$
$V_\mu(x)$	$= \bar{\psi}(x)\, \gamma_\mu\, \psi(x)$	$U^{-1} V_\mu(x)\, U$	$= a_{\mu\nu} V_\nu(l^{-1}x)$
$T_{\mu\nu}(x)$	$= \bar{\psi}(x)\, \sigma_{\mu\nu}\, \psi(x)$	$U^{-1} T_{\mu\nu}(x)\, U$	$= a_{\mu\lambda} a_{\nu\rho} T_{\lambda\rho}(l^{-1}x)$
$A_\mu(x)$	$= \bar{\psi}(x)\, \gamma_\mu \gamma_5\, \psi(x)$	$U^{-1} A_\mu(x)\, U$	$= a_{\mu\nu} A_\nu(l^{-1}x)$
$P(x)$	$= \bar{\psi}(x)\, i\gamma_5\, \psi(x)$	$U^{-1} P(x)\, U$	$= P(l^{-1}x)$

3. Stress Tensor

We first describe the conventional method of obtaining the stress tensor. Consider a Lagrange function $\mathscr{L} = \mathscr{L}(\phi_r)$, where ϕ_r describes the complete set of field variables. We introduce a nonsymmetrical stress tensor

$$\tau_{\mu\nu} = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_r)} \partial_\nu \phi_r - g_{\mu\nu} \mathscr{L}. \tag{2.49}$$

One can show that $\partial_\mu \tau_{\mu\nu} = 0$, but $\tau_{\mu\nu} \neq \tau_{\nu\mu}$ and $\partial_\nu \tau_{\mu\nu} \neq 0$. The integrals of τ_{00} and τ_{0k} are constant operators and are the correct energy-momentum operators P_μ. In order to construct a set of conserved angular momentum operators, we need a symmetric stress tensor $T_{\mu\nu}$. For a Lorentz scalar $\mathscr{L}$, we can always construct a third rank tensor $f_{\lambda\mu\nu}$, antisymmetric in $\lambda\mu$,

$$f_{\lambda\mu\nu} = -f_{\mu\lambda\nu} \tag{2.50}$$

such that

$$T_{\mu\nu} = \tau_{\mu\nu} + \partial_\lambda f_{\lambda\mu\nu} = T_{\nu\mu} \tag{2.51}$$

becomes symmetric. The added term, $\partial_\lambda f_{\lambda\mu\nu}$, preserves the conservation relation

$$\partial_\mu T_{\mu\nu} = \partial_\mu \tau_{\mu\nu} = 0. \tag{2.52}$$

It does not affect the energy-momentum operators, but will lead to the conserved angular-momentum operators $M_{\mu\nu}$,

$$M_{\mu\nu} \equiv \int d^3x \, (x_\mu T_{0\nu} - x_\nu T_{0\mu})$$

$$= \int d^3x m_{0\mu\nu}, \tag{2.53}$$

where the tensor density

$$m_{\lambda\mu\nu} \equiv x_\mu T_{\lambda\nu} - x_\nu T_{\lambda\mu} \tag{2.26}$$

is conserved, and $m_{0\mu\nu}$ is the angular momentum density.

Let us use the above method to construct the stress density for a free

vector field,

$$\mathscr{L} = -\frac{1}{4}\phi_{\mu\nu}^2 + \frac{m^2}{2}\phi_\mu^2 \tag{2.54}$$

with

$$\phi_{\mu\nu} \equiv \partial_\mu \phi_\nu - \partial_\nu \phi_\mu . \tag{2.55}$$

The nonsymmetrical stress tensor is

$$\begin{aligned} \tau_{\mu\nu} &\equiv \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_\lambda)} \partial_\nu \phi_\lambda - g_{\mu\nu}\mathscr{L} \\ &= -\phi_{\mu\lambda}\partial_\nu \phi_\lambda - g_{\mu\nu}\mathscr{L}. \end{aligned} \tag{2.56}$$

By inspection, we can obtain the symmetric tensor $T_{\mu\nu}$ by choosing

$$f_{\lambda\mu\nu} = -\phi_{\lambda\mu}\phi_\nu . \tag{2.57}$$

We then have

$$\begin{aligned} T_{\mu\nu} &= \tau_{\mu\nu} + \partial_\lambda f_{\lambda\mu\nu} \\ &= -\phi_{\mu\lambda}\partial_\nu \phi_\lambda - g_{\mu\nu}\mathscr{L} - \partial_\lambda (\phi_{\lambda\mu}\phi_\nu) \\ &= -\phi_{\mu\lambda}\phi_{\nu\lambda} + m^2\phi_\mu\phi_\nu - g_{\mu\nu}\mathscr{L} \end{aligned} \tag{2.58}$$

which is both symmetric and conserved.

We can use the action principle to give a more concise definition of the stress tensor $T_{\mu\nu}$. Consider the action

$$W = \int d^4x \, \mathscr{L}(\phi_r) \tag{2.59}$$

under a general space-time displacement

$$x_\mu \to x_\mu + \delta x_\mu . \tag{2.60}$$

We may define the stress tensor by

$$\delta W = \int d^4x\,(-\partial_\mu \delta x_\nu)\, T_{\mu\nu} \tag{2.61}$$

as a measure of the response of the system under the displacement (2.60). For a Lorentz invariant Lagrange density $\mathscr{L}$, the action is invariant under a constant rotation or a constant Lorentz transformation. Thus if $\partial_\mu \delta x_\nu$ were antisymmetrical and constant, its coefficient in δW would vanish identically. If we also impose $\partial_\mu \delta x_\nu + \partial_\nu \delta x_\mu = 0$ on the surface, we can eliminate the antisymmetric part in δW altogether. This leads immediately to a symmetric stress tensor $T_{\mu\nu}$.

Consider the variation of W in (2.59). Under the variation (2.60), we have

$$\phi(x) \to \Lambda(a)\,\phi(l^{-1}x) = \Lambda(a)\,\phi(x - \delta x) \tag{2.62}$$

with $\delta\omega_{\mu\nu}$ in a being

$$\delta\omega_{\mu\nu} = \frac{1}{2}(\partial_\nu \delta x_\mu - \partial_\mu \delta x_\nu). \tag{2.63}$$

Equations (2.62) and (2.63) provide all we need to evaluate $T_{\mu\nu}$ in (2.61).

We shall work out the $T_{\mu\nu}$ of a vector field in Eq. (2.54) explicitly. The differential form of (2.62) for a vector field is

$$\delta\phi_\mu = \frac{1}{2}(\partial_\lambda \delta x_\mu - \partial_\mu \delta x_\lambda)\,\phi_\lambda$$

$$- \delta x_\lambda \partial_\lambda \phi_\mu\,. \tag{2.64}$$

As we shall see, terms with δx_μ having no derivatives on it add up together to give

$$\int d^4x\,(-\delta x_\mu)\,\partial_\mu \mathscr{L} = \int d^4x\,(\partial_\mu \delta x_\mu)\,\mathscr{L} \tag{2.65}$$

which may also be interpreted as a change of volume d^4x due to $x \to x + \delta x$. Thus, we have

$$\delta W = \int d^4x\,[-\phi_{\mu\nu}\partial_\mu \delta\phi_\nu + m^2\phi_\mu \delta\phi_\mu]$$

$$= \int d^4x \Big[- \phi_{\mu\nu} \partial_\mu \Big(\frac{1}{2} (\partial_\lambda \delta x_\nu - \partial_\nu \delta x_\lambda) \phi_\lambda - \delta x_\lambda \partial_\lambda \phi_\nu \Big)$$

$$+ m^2 \phi_\mu \Big(\frac{1}{2} (\partial_\lambda \delta x_\mu - \partial_\mu \delta x_\lambda) \phi_\lambda - \delta x_\lambda \partial_\lambda \phi_\mu \Big) \Big]$$

$$= \int d^4x \Big[- \phi_{\mu\nu} \frac{1}{2} (\partial_\mu \partial_\lambda \delta x_\nu - \partial_\mu \partial_\nu \delta x_\lambda) \phi_\lambda$$

$$- \phi_{\mu\nu} \frac{1}{2} (\partial_\lambda \delta x_\nu - \partial_\nu \delta x_\lambda) \partial_\mu \phi_\lambda$$

$$+ \phi_{\mu\nu} (\partial_\mu \delta x_\lambda) \partial_\lambda \phi_\nu + \phi_{\mu\nu} \delta x_\lambda \partial_\mu \partial_\lambda \phi_\nu$$

$$+ \frac{m^2}{2} \phi_\mu (\partial_\lambda \delta x_\mu - \partial_\mu \delta x_\lambda) \phi_\lambda - m^2 \phi_\mu \delta x_\lambda \partial_\lambda \phi_\mu \Big]. \qquad (2.66)$$

We now look at (2.66) term by term. The first term gives

$$\phi_{\mu\nu} \frac{1}{2} (\partial_\mu \partial_\lambda \delta x_\nu - \partial_\mu \partial_\nu \delta x_\lambda) \phi_\lambda = \phi_{\mu\nu} \frac{1}{2} \partial_\mu (\partial_\lambda \delta x_\nu + \partial_\nu \delta x_\lambda) \phi_\lambda$$

$$= \frac{1}{2} \partial_\mu [(\partial_\lambda \delta x_\nu + \partial_\nu \delta x_\lambda) \phi_{\mu\nu} \phi_\lambda] \quad \text{(0 after integration)}$$

$$- \frac{1}{2} (\partial_\lambda \delta x_\nu + \partial_\nu \delta x_\lambda) \partial_\mu (\phi_{\mu\nu} \phi_\lambda)$$

$$= - \frac{1}{2} (\partial_\mu \delta x_\nu + \partial_\nu \delta x_\mu) \partial_\lambda (\phi_{\lambda\mu} \phi_\nu). \qquad (2.67)$$

We can separate the third term into a symmetric term and an antisymmetric term,

$$\phi_{\mu\nu} (\partial_\mu \delta x_\lambda) \partial_\lambda \phi_\nu = \frac{1}{2} \phi_{\mu\nu} (\partial_\mu \delta x_\nu + \partial_\lambda \delta x_\mu) \partial_\lambda \phi_\nu$$

$$+ \frac{1}{2} \phi_{\mu\nu} (\partial_\mu \delta x_\lambda - \partial_\lambda \delta x_\mu) \partial_\lambda \phi_\nu . \qquad (2.68)$$

The second term in (2.68) cancels the second term in (2.66). The fifth term in (2.66) vanishes by itself. These cancellations all correspond to the effect of pure rotations. The fourth term and the sixth term give rise to the structure as mentioned in (2.65). Thus, we can write δW as

$$\begin{aligned}\delta W &= \int d^4x \left(-\frac{1}{2}(\partial_\mu \delta x_\nu + \partial_\nu \delta x_\mu) \right) \\ &\quad \times \left[-g_{\mu\nu}\mathscr{L} - \frac{1}{2}\partial_\lambda(\phi_{\lambda\mu}\phi_\nu) - \phi_{\mu\lambda}\partial_\nu\phi_\lambda \right] \\ &= \int d^4x \left(-\frac{1}{2}(\partial_\mu \delta x_\nu + \partial_\nu \delta x_\mu) \right) T_{\mu\nu} \qquad (2.69)\end{aligned}$$

with

$$T_{\mu\nu} = -\phi_{(\mu\lambda}\partial_{\nu)}\phi_\lambda - \partial_\lambda(\phi_{\lambda(\mu}\phi_{\nu)}) - g_{\mu\nu}\mathscr{L} \qquad (2.70)$$

as desired.

As we have seen in the above derivation, if we include a nonderivative term into $\mathscr{L}$, it will not change the structure of $T_{\mu\nu}$ in (2.70). It only modifies $T_{\mu\nu}$ via the $-g_{\mu\nu}\mathscr{L}$ term. Note that a symmetrization of indices (μ, ν) in (2.70) is automatic.

4. Schwinger Relation

It is usually straightforward to verify the rotational invariance of the theory because the spatial rotations do not affect the equal-time surface $x_0 =$ constant. The invariace of the theory under a genuine Lorentz transformation is far more difficult to verify. The verification of the latter will involve the commutation relations among P_0 and M_{0k}. Schwinger found a sufficient condition for a rotational invariant theory also to be Lorentz invariant. The condition can be written as an equal-time commutation relation between the energy-density operators, $x_0 = y_0$,

$$[T_{00}(x), T_{00}(y)] = -i[T_{0k}(x) + T_{0k}(y)]\frac{\partial}{\partial x_k}\delta^3(x - y). \qquad (2.71)$$

One can verify easily from (2.71) that

$$[P_0, M_{0k}] = iP_k$$

$$[M_{0k}, M_{0l}] = -iM_{kl}.$$

To give you some feeling about the structure of (2.71), we work out the scalar field in detail. For a free scalar field, we have

$$T_{00} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\partial_k\phi)^2 + \frac{m^2}{2}\phi^2 \tag{2.72}$$

$$T_{0k} = \dot{\phi}\partial_k\phi. \tag{2.73}$$

When taking the commutator, we do not need to keep track of terms which are proportional to $\delta^3(x-y)$ because the left hand side of (2.71) is antisymmetric in x and y. This observation greatly simplifies our calculation. In fact, we only need to consider the commutator between a $\dot{\phi}$ and a $\partial\phi$ in (2.72). We then have, at $x_0 = y_0$,

$$[T_{00}(x), T_{00}(y)] = \frac{1}{4}[\dot{\phi}(x)^2, (\partial_k\phi(y))^2]$$

$$+ \frac{1}{4}[(\partial_k\phi(x))^2, \dot{\phi}(y)^2]$$

$$= i(\dot{\phi}(x)\,\partial_k\phi(y) + \dot{\phi}(y)\,\partial_k\phi(x))\frac{\partial}{\partial x_k}\delta^3(x-y)$$

$$= -i(\dot{\phi}(x)\,\partial_k\phi(x) + \dot{\phi}(y)\,\partial_k\phi(y))\frac{\partial}{\partial x_k}\delta^3(x-y)$$

$$-i[\dot{\phi}(x)\,\partial_k\phi(y) + \dot{\phi}(y)\,\partial_k\phi(x)$$

$$-\dot{\phi}(x)\,\partial_k\phi(x) - \dot{\phi}(y)\,\partial_k\phi(y)]\frac{\partial}{\partial x_k}\delta^3(x-y)$$

$$= -i\,(T_{0k}(x) + T_{0k}(y))\frac{\partial}{\partial x_k}\delta^3\,(x-y). \tag{2.74}$$

The term that we have dropped in (2.74) is of the form

$$[A(x)\,B(y) + A(y)\,B(x) - A(x)\,B(x) - A(y)\,B(x)]\frac{\partial}{\partial x_k}\delta^3(x-y)$$

$$= -(A(x) - A(y))\,(B(x) - B(y))\frac{\partial}{\partial x_k}\delta^3(x-y)$$

$$= O((x-y)^2)\frac{\partial}{\partial x_k}\delta^3(x-y) = 0. \tag{2.75}$$

Equation (2.74) is not affected if we modify T_{00} by adding an arbitrary non-derivative coupling. The inclusion of Dirac fields and vector fields does not lead to any difficulty. As long as the interaction is non-derivative, we can verify Schwinger relation straightforwardly. The verification of Lorentz invariance is no longer a Herculean task any more.

Chapter 3

DISCRETE TRANSFORMATIONS

In a quantum field theory, a discrete transformation is generated either by a unitary or an antiunitary transformation. Unlike the proper Lorentz transformations, we are not able to express these operators as functions of field variables. These operators are characterized solely by the transformation properties that they generate.

1. Parity

The parity operator P is a unitary operator in the physical Hilbert space which generates a state with space inversion. On a scalar field $\phi(x)$ and a vector field $\phi_\mu(x)$, the operator P leads to

$$P^{-1}\phi(x)\,P = \eta_P\,\phi(x') \tag{3.1}$$

with

$$\eta_P(\phi) = \pm 1 \tag{3.2}$$

$$(x')_\mu = (x_{0'} - x_k) \tag{3.3}$$

and

$$P^{-1}\phi_\mu(x)\,P = \eta_P\,(\phi_0(x'), -\phi_k(x'))$$

$$= \eta_P(\phi_P)\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}(\phi_\mu(x')) \tag{3.4}$$

with

$$\eta_P(\phi_\mu) = \pm 1. \tag{3.5}$$

The η_P describes the intrinsic parity of the field. We call a scalar field $\phi(x)$ a true scalar field if $\eta_P(\phi) = 1$, and a pseudo scalar field if $\eta_P(\phi) = -1$. We call a vector field $\phi_\mu(x)$ a polar vector field if $\eta_P(\phi_\mu) = 1$, and an axial vector field if $\eta_P(\phi_\mu) = -1$.

The transformation law for a Dirac field is

$$P^{-1}\,\psi(x)\,P = \eta_P\,\gamma_0\,\psi(x') \tag{3.6a}$$

$$P^{-1}\,\psi^\dagger(x)\,P = \eta_P^*\,\psi^\dagger(x')\,\gamma_0 \tag{3.6b}$$

with

$$\eta_P(\psi) = \pm 1, \pm i. \tag{3.7}$$

Both the free Lagrange functions and the quantization relations for ϕ, ψ, and ϕ_μ are invariant under the parity transformations (3.1)–(3.7).

Note that P^2 transforms coordinate x_μ into x_μ. Thus, P^2 is either an identity operator or a rotation of 2π. For scalar and vector fields, the rotation of 2π is equivalent to no rotations and hence $\eta_P^2 = 1$. For a Dirac field, ψ transforms as a spinor, and a rotation of 2π leads to $-\psi$. Thus, η_P^2 is either 1 or -1 which implies Eq. (3.7).

The phase factor η_P of a Dirac field ψ never enters into the transformation laws of the bilinear forms. For instance, the transformation laws for $\bar{\psi}\gamma_\mu\psi$ and $\bar{\psi}\gamma_\mu\gamma_5\psi$ are

$$\begin{aligned} P^{-1}\,\bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,P &= P^{-1}\,\psi^\dagger(x)\,P\,\gamma_0\,\gamma_\mu\,P^{-1}\,\psi(x)\,P \\ &= \psi^\dagger(x')\,\gamma_0\,\gamma_0\,\gamma_\mu\,\gamma_0\,\psi(x') \\ &= \bar{\psi}(x')\begin{pmatrix}\gamma_0 \\ & -\gamma_k\end{pmatrix}\psi(x'), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &P^{-1}\,\bar{\psi}(x)\,\gamma_\mu\,\gamma_5\,\psi(x)\,P \\ &= \psi^\dagger(x')\,\gamma_0\,\gamma_0\,\gamma_\mu\,\gamma_5\,\gamma_0\,\psi(x') \\ &= \bar{\psi}(x')\begin{pmatrix}-\gamma_0 \\ & \gamma_k\end{pmatrix}\gamma_5\,\psi(x'), \end{aligned} \tag{3.9}$$

with

$$(x')_\mu = (x_0, -x_k). \tag{3.10}$$

Thus, $\bar{\psi}\gamma_\mu\psi$ transforms as a polar vector, and $\bar{\psi}\gamma_\mu\gamma_5\psi$ transforms as an axial vector. We summarize the transformation laws for other bilinear form in Table II.

Table II Transformation properties of bilinear forms under the parity transformation P. All the bilinear forms are hermitian by construction.

Bilinear Forms	Transformation Laws	Remark
$S(x) \equiv \bar{\psi}(x)\,\psi(x)$	$P^{-1}S(x)\,P = S(x')$	scalar
$V_\mu(x) \equiv \bar{\psi}(x)\,\gamma_\mu\psi(x)$	$P^{-1}V_\mu(x)\,P = (V_0(x'), -V_k(x'))$	Polar Vector
$T_{\mu\nu}(x) \equiv \bar{\psi}(x)\,\sigma_{\mu\nu}\psi(x)$	$P^{-1}T_{\mu\nu}(x)\,P = (-T_{0k}(x'), T_{lm}(x'))$	antisymmetric tensor
$A_\mu(x) \equiv \bar{\psi}(x)\,\gamma_\mu\gamma_5\,\psi(x)$	$P^{-1}A_\mu(x)\,P = (-A_0(x'), A_k(x'))$	axial vector
$P(x) \equiv \bar{\psi}(x)\,i\gamma_5\psi(x)$	$P^{-1}\,P(x)\,P = -P(x')$	pseudoscalar

By a proper choice of phase $\eta_P(\phi_\mu)$, we can make either $\phi_\mu\bar{\psi}\gamma_\mu\psi$ or $\phi_\mu\bar{\psi}\gamma_\mu\gamma_5\psi$ a true scalar, but not both simultaneously. Thus, a combination of the form

$$\phi_\mu\,\bar{\psi}\,\gamma_\mu\,(a + b\gamma_5)\,\psi$$

can never be invariant under parity if $ab \neq 0$.

2. Charge Conjugation

As we have learned in the first part of this volume, we need twice the number of fields to describe a charged field. The Lagrange function for a free massive charged scalar field is

$$\mathcal{L} = \sum_{i=1,2}\left[\frac{1}{2}(\partial_\mu\phi^{(i)})^2 - \frac{m^2}{2}(\phi^{(i)})^2\right]$$

$$= \partial_\mu\phi^\dagger\,\partial_\mu\phi - m^2\phi^\dagger\phi \tag{3.11}$$

where (ϕ^1, ϕ^2) are hermitian (real) fields, and

$$\phi \equiv \frac{\phi^1 + i\phi^2}{\sqrt{2}} \tag{3.12}$$

$$\phi^\dagger \equiv \frac{\phi^1 - i\phi^2}{\sqrt{2}}. \tag{3.13}$$

It is simple to see that a Dirac field is a charged field. We can also construct a charged vector field by introducing a complex vector field ϕ_μ and the Lagrange function,

$$\mathscr{L} = -\frac{1}{2}\phi^\dagger_{\mu\nu}\phi_{\mu\nu} + m^2\phi^\dagger_\mu\phi_\mu \tag{3.14}$$

with

$$\phi_{\mu\nu} \equiv \partial_\mu\phi_\nu - \partial_\nu\phi_\mu \tag{3.15}$$

$$\phi^\dagger_{\mu\nu} \equiv \partial_\mu\phi^\dagger_\nu - \partial_\nu\phi^\dagger_\mu. \tag{3.16}$$

Basically, a charge conjugation operation transforms a complex field ϕ into its complex conjugation $\phi^\dagger$. The free Lagrange functions are invariant under such a transformation.

The charge conjugation properties for a charged scalar and vector fields are simply

$$C^{-1}\phi(x)C = \eta_c(\phi)\phi^\dagger(x), \qquad \eta_c(\phi) = \pm 1 \tag{3.17}$$

$$C^{-1}\phi_\mu(x)C = \eta_c(\phi_\mu)\phi^\dagger_\mu, \qquad \eta_c(\phi_\mu) = \pm 1. \tag{3.18}$$

Note that the coordinate x is not affected by C. The hermitian fields ϕ^1 and ϕ^2 transform according to

$$C^{-1}\phi^1(x)C = \eta_c\phi^1(x) \tag{3.19a}$$

$$C^{-1}\phi^2(x)C = -\eta_c\phi^2(x). \tag{3.19b}$$

In other words, the charge conjugation transformation is a reflection in the $\phi^1\phi^2$ space.

The transformation law for a Dirac field ψ is more complicated, and is representation dependent. In the Majorana representation specified by

$$\gamma_0 = \text{imaginary and antisymmetric}$$

$$\gamma = \text{imaginary and symmetric,}$$

the transformation law of ψ under C takes a simple form

$$C^{-1}\,\psi(x)\,C = \eta_c(\psi)\,\psi^\dagger(x) \tag{3.20}$$

$$C^{-1}\,\psi^\dagger(x)\,C = \eta_c^*(\psi)\,\psi(x). \tag{3.21}$$

It is straightforward to verify that the free Lagrange functions are indeed invariant under the transformation (3.17)–(3.21).

We can work out the transformation laws for all bilinear combinations of $\bar{\psi}$ and ψ. The phases $\eta_c(\psi)$ and $\eta_c(\psi)^*$ cancel each other, and the transformation laws do not depend on $\eta_c(\psi)$ at all. For instance, the transformation law for $\bar{\psi}(x)\,\psi(x)$ is

$$\begin{aligned} C^{-1}\,\bar{\psi}(x)\,\psi(x)\,C &= C^{-1}\,\psi_\alpha^\dagger(x)\,(\gamma_0)_{\alpha\beta}\,\psi_\beta(x)\,C \\ &= C^{-1}\,\psi_\alpha^\dagger(x)\,C\,(\gamma_0)_{\alpha\beta}\,C^{-1}\psi_\beta(x)\,C \\ &= \psi_\alpha(x)\,(\gamma_0)_{\alpha\beta}\,\psi_\beta^\dagger(x) \\ &= -\psi_\beta^\dagger(x)\,(\gamma_0)_{\alpha\beta}\,\psi_\alpha(x) \\ &= \psi_\beta^\dagger(x)\,(\gamma_0)_{\beta\alpha}\,\psi_\alpha(x) \\ &= \bar{\psi}(x)\,\psi(x). \end{aligned} \tag{3.22}$$

In (3.22), we have used the fact that ψ and $\psi^\dagger$ are anticommuting, and that $(\gamma_0)_{\alpha\beta}$ is antisymmetric. To avoid any extra contribution when we anticommute ψ and $\psi^\dagger$, we may assume that the operators are normal-ordered. We summarize the transformation properties of other bilinear combinations in Table III.

Note that the vector current $\bar{\psi}\gamma_\mu\psi$ is odd under C and that the axial current $\bar{\psi}\gamma_\mu\gamma_5\psi$ is even under C. Thus, a term such as $\bar{\psi}\gamma_\mu(1\pm\gamma_5)\psi\phi_\mu$ always violates C-parity. We call a real spinor field obeying $\psi^\dagger = \psi$ a Majorana spinor.

Table III Transformation properties of bilinear forms under the charge conjugation transformation C. Note that the coordinate x is not affected by C.

Billinear Forms	Transformation Laws
$S(x) \equiv \bar{\psi}(x)\,\psi(x)$	$C^{-1}S(x)\,C = S(x)$
$V_\mu(x) \equiv \bar{\psi}(x)\,\gamma_\mu\psi(x)$	$C^{-1}V_\mu(x)\,C = -V_\mu(x)$
$T_{\mu\nu}(x) \equiv \bar{\psi}(x)\,\sigma_{\mu\nu}\psi(x)$	$C^{-1}T_{\mu\nu}(x)\,C = -T_{\mu\nu}(x)$
$A_\mu(x) \equiv \bar{\psi}(x)\,\gamma_\mu\gamma_5\,\psi(x)$	$C^{-1}A_\mu(x)\,C = A_\mu(x)$
$P(x) \equiv \bar{\psi}(x)\,i\gamma_5\psi(x)$	$C^{-1}P(x)\,C = P(x)$

3. Time Reversal

The time reversal transformation is represented by an antiunitary operator T in the quantum mechanical Hilbert space. By an antiunitary operator, we mean:

(1) It is a unitary operator

$$T^{-1} = T^{\dagger}. \tag{3.23}$$

(2) It is an antilinear operator

$$T(\lambda_1 |\,1\rangle + \lambda_2 |\,2\rangle) = \lambda_1^* T|\,1\rangle + \lambda_2^* T|\,2\rangle. \tag{3.24}$$

(3) If

$$|\,p'\rangle = T|\,p\rangle, \text{ and } \langle\, q'| = \langle\, q|\,T^{-1},$$

then

$$\langle\, q'|\,p'\rangle = (\langle\, q|\,p\rangle)^*. \tag{3.25}$$

Condition (3) actually follows from condition (2). Equations (3.24) and (3.25) lead to the following relations

$$T\lambda = \lambda^* T \tag{3.26}$$

$$(\langle\, a|\,T)|\,b\rangle = [\,\langle\, a|(T\,b\rangle|\,)\,]^*. \tag{3.27}$$

We can see easily that we need an antiunitary operator T to describe the

time reversal transformation. Consider the transformation of a scalar field $\phi(x, t)$ and its conjugate momentum variable $\dot{\phi}(x, t)$ under T

$$T^{-1}\,\phi(x)\,T = \eta_T\,\phi(x') \tag{3.28}$$

$$T^{-1}\,\dot{\phi}(x)\,T = -\,\eta_T\,\dot{\phi}(x') \tag{3.29}$$

where

$$(x')_\mu = (-x_0, x_k) \tag{3.30}$$

and the phase

$$\eta_T = \pm 1. \tag{3.31}$$

The transformation on the canonical quantization relation,

$$[\phi(x), \dot{\phi}(y)] = i\delta^3(x-y) \tag{3.32}$$

gives

$$[\phi(x'), -\dot{\phi}(y')] = T^{-1}\,iT\,\delta^3(x'-y'). \tag{3.33}$$

Equation (3.33) gives rise to the correct canonical quantization relation only if T is an antiunitary operator.

The transformation laws for the vector and the Dirac fields in the Majorana representation are

$$T^{-1}\,\phi_\mu(x)\,T = \eta_T\,(-\phi_0(x'), \phi_k(x')) \tag{3.34}$$

with

$$\eta_T\,(\phi_\mu) = \pm 1 \tag{3.35}$$

and

$$T^{-1}\,\psi(x)\,T = \eta_T\,i\gamma_0\gamma_5\,\psi(x') \tag{3.36}$$

$$T^{-1}\,\psi^\dagger(x)\,T = \eta_T^*\,\psi^\dagger(x')i\gamma_0\gamma_5 \tag{3.37}$$

with

$$\eta_T(\psi) = \pm 1, \pm i. \tag{3.38}$$

One can show that these transformations leave the free Lagrange functions, field equations, and the canonical quantization relations invariant.

Just as in the other discrete transformations, we can work out the transformation laws for all the bilinear forms $\bar{\psi}\Gamma\psi$. In the Majorana representation, all γ_μ's have the same reality condition. (γ_μ is purely imaginary.) This enables us to work out the transformation laws straightforwardly. For instance, the transformation for $\bar{\psi}(x)\,\psi(x)$ is

$$\begin{aligned} T^{-1}\,\bar{\psi}(x)\,\psi(x)\,T &= T^{-1}\,\psi^\dagger(x)\,\gamma_0\,\psi(x)\,T \\ &= T^{-1}\,\psi^\dagger(x)\,T\,T^{-1}\gamma_0\,T\,T^{-1}\,\psi(x)\,T \\ &= \psi^\dagger(x')\,i\gamma_0\gamma_5\,(-\gamma_0)\,i\gamma_0\gamma_5\,\psi(x') \\ &= \bar{\psi}(x')\,\psi(x') \end{aligned} \tag{3.39}$$

where we have used the fact that γ_μ is imaginary and hence

$$T^{-1}\gamma_\mu\,T = -\,\gamma_\mu. \tag{3.40}$$

We summarize the transformation laws for all the bilinear forms in Table IV. It is interesting to see that $V_\mu(x)$ and $A_\mu(x)$ have the same transformation law under T. Hence, with a proper choice of phases, the interaction

Table IV Transformation properties of bilinear forms under the time reversal transformation T. The time-reversal coordinates are $x'_\mu = (-x_0, x_k)$.

Bilinear Forms	Transformation Laws
$S(x) \equiv \bar{\psi}(x)\,\psi(x)$	$T^{-1}S(x)\,T = S(x')$
$V_\mu(x) \equiv \bar{\psi}(x)\,\gamma_\mu\psi(x)$	$T^{-1}V_\mu(x)\,T = (V_0(x'), -V_k(x'))$
$T_{\mu\nu}(x) \equiv \bar{\psi}(x)\,\sigma_{\mu\nu}\psi(x)$	$T^{-1}T_{\mu\nu}(x)\,T = (T_{0k}(x'), -T_{lm}(x'))$
$A_\mu(x) \equiv \bar{\psi}(x)\,\gamma_\mu\gamma_5\,\psi(x)$	$T^{-1}A_\mu(x)\,T = (A_0(x'), -A_k(x'))$
$P(x) \equiv \bar{\psi}(x)\,i\gamma_5\psi(x)$	$T^{-1}\,P(x)\,T = -\,P(x')$

$$\phi_\mu\,\bar{\psi}(a\,\gamma_\mu + b\,\gamma_\mu\,\gamma_5)\psi, \qquad a, b \text{ real}$$

is invariant under T.

4. TCP Theorem

Even though the free Lagrange functions are invariant under P, C and T, an interaction can violate any or all of these discrete symmetries. Thus, it is pleasing to know that a local, Lorentz invariant field theory is always invariant under the combined operation of TCP. Just as T, TCP operator is antiunitary. We can obtain the effect of TCP on the fields ϕ, ϕ_μ and ψ by operating T, C and P on them in sequence. The combined transformation gives

$$(\mathrm{TCP})^{-1}\,\phi(x)\,(\mathrm{TCP}) = \eta_{\mathrm{TCP}}(\phi)\,\phi^\dagger(x') \tag{3.41}$$

$$(\mathrm{TCP})^{-1}\,\phi_\mu(x)\,(\mathrm{TCP}) = \eta_{\mathrm{TCP}}(\phi_\mu)\,(-\phi_\mu^\dagger(x')) \tag{3.42}$$

$$(\mathrm{TCP})^{-1}\,\psi(x)\,(\mathrm{TCP}) = \eta_{\mathrm{TCP}}(\psi)\,\psi^\dagger(x')\,(-i\gamma_5) \tag{3.43}$$

$$(\mathrm{TCP})^{-1}\,\psi^\dagger(x)\,(\mathrm{TCP}) = \eta^*_{\mathrm{TCP}}(\psi)\,i\gamma_5\,\psi(x') \tag{3.44}$$

where

$$(x')_\mu = (-x_0, -x_k) = -x_\mu. \tag{3.45}$$

The phases associated with the TCP transformation is related trivially to the product of the individual phases associated with T, C and P. It is important to note that all components of ϕ_μ transform in the same way.

We can work out the transformation laws for all the bilinear combinations. Their transformation laws are

$$(\mathrm{TCP})^{-1}\,\bar\psi(x)\,\psi(x)\,(\mathrm{TCP}) = \bar\psi(x')\,\psi(x') \tag{3.46a}$$

$$(\mathrm{TCP})^{-1}\,\bar\psi(x)\,\gamma_\mu\,\psi(x)\,(\mathrm{TCP}) = -\bar\psi(x')\,\gamma_\mu\,\psi(x') \tag{3.46b}$$

$$(\mathrm{TCP})^{-1}\,\bar\psi(x)\,\sigma_{\mu\nu}\,\psi(x)\,(\mathrm{TCP}) = \bar\psi(x')\,\sigma_{\mu\nu}\,\psi(x') \tag{3.46c}$$

$$(\mathrm{TCP})^{-1}\,\bar\psi(x)\,\gamma_\mu\gamma_5\,\psi(x)\,(\mathrm{TCP}) = -\bar\psi(x')\,\gamma_\mu\gamma_5\,\psi(x') \tag{3.46d}$$

$$(\mathrm{TCP})^{-1}\,\bar\psi(x)\,i\gamma_5\,\psi(x)\,(\mathrm{TCP}) = \bar\psi(x')\,i\gamma_5\,\psi(x') \tag{3.46e}$$

where all the bilinear forms in (3.46) are hermitian by construction. Equation (3.46) reveals that the transformation laws of $\bar\psi\Gamma\psi$ depend only

on the rank of the tensor field. The above result remains valid even after we include the derivative ∂_μ into the construction. Thus, we have

$$(\mathrm{TCP})^{-1}\,\bar{\psi}\,\Gamma_{\mu_1\cdots\mu_n}\,\psi(x)\,(\mathrm{TCP}) = (-1)^n\left(\bar{\psi}(x')\,\Gamma_{\mu_1\cdots\mu_n}\,\psi(x')\right)^\dagger \tag{3.47}$$

where the adjoint (†) operations are omitted in (3.46) because the bilinear forms are hermitian. By choosing the phases in (3.41) and (3.42) to be 1, we also have

$$(\mathrm{TCP})^{-1}\,\phi_{\mu_1\cdots\mu_n}(x)\,(\mathrm{TCP}) = (-1)^n\left(\phi_{\mu_1\cdots\mu_n}(x')\right)^\dagger. \tag{3.48}$$

Thus, we have a universal transformation law for all local fields and $\bar{\psi}\Gamma\psi$. For an arbitrary hermitian, local, and Lorentz invariant Lagrange function $\mathscr{L}(x)$, we always have

$$(\mathrm{TCP})^{-1}\,\mathscr{L}(x)\,(\mathrm{TCP}) = \mathscr{L}(x'). \tag{3.49}$$

We can show that the field equations and the quantization rules are also invariant under TCP. The discrete transformation TCP is an exact symmetry. This is known as the TCP theorem.

The proof of TCP theorem depends also on the quantization rules known as the connection between the spin and the statistics: We quantize the integer spin fields by commutator relations (Bose statistics) and the half (plus an) integer spin fields by anticommutator relations (Fermi statistics).

TCP theorem implies that (1) For stable particles, a particle and its antiparticle should have the same mass, (2) For unstable particles, they should also have equal decay rate (or life time). However, the decay rates for individual channels may not be equal between a particle and its antiparticle.

5. Spin and Statistics

Pauli first established the connection between the spin and the statistics by ruling out the wrong connections. We can summarize his investigation as follows: It is not possible to quantize consistently an integer spin system by Fermi statistics and a half-integer system by Bose statistics.

The general strategy of the proof goes as follows. For a half-integer spin system, such as a free spin 1/2 Dirac field, the system contains negative energy solutions. If we quantize the system by Bose statistics, then there is no exclusion principle to help us to fill the negative energy sea. Then, the Hamiltonian is no longer bounded from below.

To show that it is not possible to quantize an integral spin system such as a free scalar field by Fermi statistics, we note that both expressions,

$$\langle 0|[\phi_i(x), \phi_j(x')]|0\rangle$$

and

$$\langle 0|\{\phi_i(x), \phi_j(x')\}|0\rangle$$

obey Klein-Gordon equation. For each pair of ϕ_i and ϕ_j, they provide two linearly independent solutions to the Klein-Gordon equation. It is easy to see that when we quantize ϕ and $\dot{\phi}$ according to fermi statistics, we must have at $x_0 = y_0$,

$$\{\phi_i(x), \dot{\phi}_j(y)\} = C_{ij}\,\delta^3(x-y) \tag{3.50a}$$

$$\{\phi_i(x), \phi_j(y)\} = \{\dot{\phi}_i(x), \dot{\phi}_j(y)\} = 0 \tag{3.50b}$$

where $C_{ij}(\partial)$ may contain a finite number of derivatives. We can construct the two independent solutions of Klein-Gordon equations as

$$\Delta(x-y) = -i\int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik\cdot(x-y)} - e^{ik(x-y)})$$

and

$$\Delta_1(x-y) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}(e^{-ik\cdot(x-y)} + e^{ik(x-y)}).$$

The solution $\Delta(x-y)$ is antisymmetric (odd) in $x-y$ and vanishes for spacelike separation. The even function Δ_1 does not vanish at spacelike separation, and falls off exponentially as e^{-mr},

$$\Delta_1(x-y) = \frac{1}{4\pi r}\frac{\partial}{\partial r}\begin{cases} Y_0\left(m\sqrt{t^2-r^2}\right), & |t|>|r| \\ -\dfrac{2}{\pi}K_0\left(m\sqrt{r^2-t^2}\right), & |r|>|t|. \end{cases}$$

Thus, if $\langle 0|\{\phi_i(x), \phi_j(y)\}|0\rangle$ vanishes for spacelike separation, it must be proportional to $\Delta(x-y)$. Then, the other combination $\langle 0|[\phi_i(x),$

$\phi_j(y)]\,|\,0\rangle$ should contain $\Delta_1(x-y)$, which can no longer vanish at a finite spacelike separation. Since mesons (integer spin particles) may be produced singly, we expect that the emissions of two mesons with a spacelike separation are dynamically independent. Thus, quantization rules (3.50) are inconsistent with causality. We can extend the above result to systems with arbitrary integral spins.

We wish to add that the total symmetrization and total antisymmetrization are only two possible representations of the permutation group. In general, there also exist other higher dimensional representations which can lead to "intermediate" or "para" statistics. Some physicists proposed that quarks actually obey parastatistics. Only composite objects such as mesons and baryons have normal statistics.

Chapter 4

PATH INTEGRAL FORMULATION

1. Non-relativistic Quantum Mechanics

We begin with a non-relativistic quantum mechanics system. Consider the transition amplitude for a particle from the position x_a at $t = t_a$ to the position x_b at $t = t_b$. Classically, there is only one (occasionally, a few) path which passes through x_a and x_b at t_a and t_b. Quantum-mechanically, however, there are an infinite number of paths which contribute. A different path contributes to a different amplitude and a different phase. The resultant amplitude is a coherent sum over amplitudes associated with all possible paths. (See Fig. 4.1)

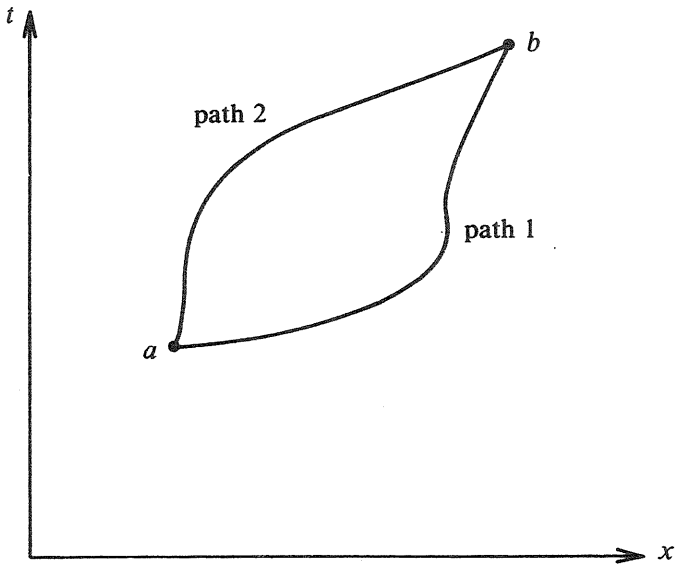


Fig. 4.1 Two possible paths joining points a to b.

Consider a non-relativistic problem

$$L = \frac{1}{2} m \dot{x}^2 - V(x) \tag{4.1}$$

$$H = \frac{p^2}{2m} + V(x). \tag{4.2}$$

The transition amplitude is

$$K(b, a) \equiv \langle x_b, t_b | x_a, t_a \rangle$$

$$= \langle x_b | e^{-iH(t_b - t_a)} | x_a \rangle. \tag{4.3}$$

Introducing a complete set of states at the sequence of times

$$t_b \equiv t_N > t_{N-1} > t_{N-2} > \cdots > t_1 > t_0 \equiv t_a,$$

we have

$$\langle b | a \rangle = \int dx_1 \cdots dx_{N-1} \langle b | x_{N-1}, t_{N-1} \rangle$$

$$\times \langle x_{N-1}, t_{N-1} | x_{N-2}, t_{N-2} \rangle \cdots \langle x_2, t_2 | x_1, t_1 \rangle \langle x_1, t_1 | a \rangle. \tag{4.4}$$

Considering $t_2 - t_1$ small and assuming $V = 0$ for the moment, we have

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int \frac{dk}{(2\pi)} \langle x_2, t_2 | k \rangle \langle k | x_1, t_1 \rangle$$

$$= \int \frac{dk}{2\pi} e^{-\frac{ik^2}{2m}(t_2 - t_1) + ik(x_2 - x_1)}$$

$$= \sqrt{\frac{m}{i2\pi(t_2 - t_1)}} e^{\frac{im(x_2 - x_1)^2}{2(t_2 - t_1)}}$$

$$= \sqrt{\frac{m}{i2\pi(t_2 - t_1)}} e^{\frac{im}{2}\left(\frac{x_2 - x_1}{t_2 - t_1}\right)^2 (t_2 - t_1)}$$

$$= \sqrt{\frac{m}{i2\pi(t_2 - t_1)}} e^{i\int_{t_1}^{t_2} dt \frac{m}{2}\dot{x}^2}. \tag{4.5}$$

For $V \neq 0$ and with small $t_2 - t_1$, we expect $x_2 - x_1$ to be small, and can treat $V(x)$ as a constant. Then we have

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int \frac{dk}{2\pi} e^{-i\left(\frac{k^2}{2m} + V\right)(t_2 - t_1) + ik(x_2 - x_1)}$$

$$= \sqrt{\frac{m}{i2\pi(t_2 - t_1)}}\, e^{i\int_{t_1}^{t_2} d\tau\, L(x(\tau))} \tag{4.6}$$

where

$$L(x(\tau)) = \frac{m}{2}\dot{x}^2 - V(x). \tag{4.7}$$

After inserting (3.6) back into (3.4), we obtain

$$\langle b | a \rangle = \int dx_1 dx_2 \cdots dx_{N-1} \prod_i \sqrt{\frac{m}{i2\pi(t_i - t_{i-1})}}$$

$$\times\, e^{i\int_{t_a}^{t_b} dt\, L(x(\tau))}. \tag{4.8}$$

Instead of interpreting (4.8) as multiple integrals, we view the integrand as a functional of the path $x(\tau)$ and interpret (4.8) as an integral over all paths. In other words, we refer this expression as a path integral. Writing

$$\int dx \cdots dx_{N-1} \prod_i \sqrt{\frac{m}{i2\pi(t_i - t_{i-1})}} \text{ as } \int \mathscr{D}\{x(t)\}$$

and denoting it as the symbol of a path integral, we have

$$\langle b | a \rangle = \int \mathscr{D}\{x(\tau)\}\, e^{i\int_{t_a}^{t_b} d\tau\, L(x(\tau))}. \tag{4.9}$$

2. Classical Limit

If we include $\hbar$ in our expansion, we have

$$\langle b | a \rangle = \int \mathscr{D}\{x(\tau)\}\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau\, L(x(\tau))}. \tag{4.10}$$

As $\hbar \to 0$, we can use the stationary phase approximation to evaluate the path integration along the classical trajectory. We can determine the

classical path $x = x_c(t)$ by

$$\delta \int_{t_a}^{t_b} dt\, L(x_c(t_1)) = 0. \tag{4.11}$$

Expanding around the classical path, we have

$$\frac{i}{\hbar}\int_a^b d\tau\, L(x(\tau)) = \frac{i}{\hbar}\int_a^b d\tau\, L(x_c(\tau))$$

$$+ \frac{i}{2\hbar}\int_{t_a}^{t_b} d\tau\,[m(\delta\dot{x})^2 - V''(x_0)(\delta x)^2] + O((\delta x)^3). \tag{4.12}$$

Note that the terms linear in $\delta x \equiv x(\tau) - x_c(\tau)$ drop out in (4.12). Ignoring $O((\delta x)^3)$ terms or higher, we have

$$\langle b \mid a \rangle = e^{iS_{cl}/\hbar} \int \mathscr{D}\{x(\tau)\}\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau \left[\frac{m}{2}(\delta\dot{x})^2 - \frac{V''(x_0)}{2}(\delta x)^2\right]} \tag{4.13}$$

where S_{cl} is the classical action. In terms of the deviation y,

$$y(\tau) \equiv x(\tau) - x_c(\tau) = \delta x \tag{4.14}$$

$$y(t_a) = y(t_b) = 0, \tag{4.15}$$

we have

$$\langle b \mid a \rangle = e^{iS_{cl}/\hbar} \int \mathscr{D}\{y(\tau)\}\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau \left[\frac{m}{2}\dot{y}^2 - \frac{1}{2}V''(x_0)\,y^2\right]}. \tag{4.16}$$

The y-integration is a Gaussian integral. We can formally do the y-integration and lead to a determinant.

We can work out the path integral for the harmonic oscillator exactly. Consider the harmonic oscillator

$$L = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2 \tag{4.17}$$

$$\langle b \mid a \rangle = \int \mathscr{D}\{x(\tau)\}\, e^{\frac{i}{\hbar}\int_{t_a}^{t_b} d\tau\, L(x(\tau))}. \tag{4.18}$$

Since the action is a quadratic function of x and $\dot{x}$, we can apply the above method here without introducing any approximation. The classical path is

$$\delta \int d\tau\, L(x(\tau)) = 0 \tag{4.19}$$

$$\ddot{x}_c + \omega^2 x_c = 0 \tag{4.20}$$

with

$$x_c(t_a) = x_a,\ x_c(t_b) = x_b. \tag{4.21}$$

Equations (4.20) and (4.21) specify the trajectory uniquely. Defining

$$y(\tau) \equiv x(\tau) - x_c(\tau)$$

as before, we have

$$\begin{aligned}
\langle b \mid a \rangle = & \int \mathscr{D}\{x(\tau)\} \\
& \times \exp \frac{i}{\hbar} \left\{ \int d\tau \left[\frac{m}{2} (\dot{y}+\dot{x}_c)^2 - \frac{m}{2} \omega^2 (y+x_c)^2 \right] \right\} \\
= & \int \mathscr{D}\{y(\tau)\} \exp \frac{i}{\hbar} \left\{ \int d\tau \left(\frac{m}{2} \dot{x}_c^2 - \frac{m\omega^2}{2} x_c^2 \right) \right. \\
& + \int d\tau\, (-y)(m\ddot{x}_c + m\omega^2 x_c) \\
& \left. + \int d\tau \left(\frac{m}{2} \dot{y}^2 - \frac{m}{2} \omega^2 y^2 \right) \right\}.
\end{aligned} \tag{4.22}$$

Note that the cross terms vanish identically due to the classical equation (Eq. (4.20)). It is obvious that the last factor in (4.22) is independent of the end points x_a, x_b. Hence, we have

$$\langle b \mid a \rangle = e^{\frac{i}{\hbar} S_{\mathrm{cl}}} \int \mathscr{D}\{y(\tau)\}\, e^{\frac{i}{\hbar} \int_{t_a}^{t_b} d\tau \left(\frac{m}{2} \dot{y}^2 - \frac{m\omega^2}{2} y^2 \right)}$$

$$= e^{\frac{i}{\hbar} S_{\mathrm{cl}}(b,a)} F(t_b - t_a), \tag{4.23}$$

with

$$F(t_b - t_a) \equiv \int \mathscr{D}\{y(\tau)\}\, e^{\frac{i}{\hbar}\int_{t_b}^{t_a} d\tau \left(\frac{m}{2}\dot{y}^2 - \frac{m}{2}\omega^2 y^2\right)} \tag{4.24}$$

$$y(t_a) = y(t_b) = 0. \tag{4.25}$$

The classical action is

$$S_{\mathrm{cl}}(b,a) = \frac{m\omega}{2 \sin \omega(t_b - t_a)} [(x_a^2 + x_b^2) \cos \omega(t_b - t_a) - 2 x_a x_b]. \tag{4.26}$$

We may use the unitarity condition of $\langle b \mid a \rangle$ to determine $F(t_b - t_a)$. In the following, we shall compute the ground state to ground state transition matrix element through which we also determine $F(t_b - t_a)$.

The ground state to ground state transition matrix is

$$\langle 0, t_b \mid 0, t_a \rangle = \int dx_a\, dx_b \langle 0 \mid x_b \rangle \langle b \mid a \rangle \langle x_a \mid 0 \rangle. \tag{4.27}$$

We can either look up the ground state wave function $\langle x \mid 0 \rangle$ from any elementary quantum mechanics text book, or obtain it by using the following trick. We consider the large Euclidean time dependence, $t_b - t_a \to -i\tau$. Only the ground state contributes to the large τ limit, giving

$$\lim_{\tau\to\infty} \langle b \mid a \rangle = \lim_{\tau\to\infty} \exp\left[\frac{i}{\hbar} \frac{m\omega}{-2i \sinh \omega\tau} (x_a^2 + x_b^2) \cosh \omega\tau\right] F(\tau)$$

$$= e^{-\frac{m\omega x_a^2}{2\hbar}} F(\tau)\, e^{-\frac{m\omega x_b^2}{2\hbar}}$$

$$= \langle x_b \mid 0 \rangle\, e^{-E_0 \tau} \langle 0 \mid x_a \rangle.$$

We can identify the properly normalized ground state wave function easily as

$$\langle x \mid 0 \rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}. \tag{4.28}$$

Substituting (4.28) into (4.27), we have

$$\langle 0, t_b | 0, t_a \rangle = \int dx_b dx_a \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x_a^2/2\hbar}$$

$$\times e^{\frac{i}{\hbar} S_{\mathrm{cl}}(b,a)} F(t_b - t_a) \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x_b^2/2\hbar}$$

$$= F(t_b - t_a) \sqrt{\frac{m\omega}{\pi\hbar}} \int dx_b\, dx_a$$

$$\times \exp\left\{-\frac{m\omega}{2\hbar}\left[(1 - i\cot\omega(t_b - t_a))\, x_a^2\right.\right.$$

$$\left.\left. + (1 - i\cot\omega(t_b - t_a))\, x_b^2 + \frac{i\, 2x_a x_b}{\sin\omega(t_b - t_a)}\right]\right\}. \tag{4.29}$$

It is straightforward to work out the Gaussian integral (4.29), giving

$$\langle 0, t_b | 0, t_a \rangle = F(t_a - t_b) \sqrt{\frac{m\omega}{\pi\hbar}} \frac{\pi}{\frac{m\omega}{2\hbar}\left[(1 - i\cot\omega(t_b - t_a))^2 + (\sin\omega(t_b - t_a))^{-2}\right]^{1/2}}$$

$$= e^{-i\omega\frac{(t_b - t_a)}{2}} F(t_b - t_a) \sqrt{\frac{2\pi i\hbar \sin\omega(t_b - t_a)}{m\omega}}. \tag{4.30}$$

Equation (4.30) gives the familiar result $\exp\left[-\frac{i}{2}\omega(t_b - t_a)\right]$ provided that

$$F(T) = \left(\frac{m\omega}{2\pi i\hbar \sin\omega T}\right)^{1/2}. \tag{4.31}$$

In (4.31), we have assigned the classical ground state energy as $E_0 = 0$. We may assign the ground state energy as E_0 which will modify $F(T)$ by a factor $e^{-iE_0 T}$. In particular, it is more convenient to absorb the zero point energy in E_0 and redefine the quantum ground state energy as zero. Then, we have $\langle 0, t_a | 0, t_b \rangle = 1$.

Similary, we can work out the transition amplitude in the presence of an external force $f(t)$. The Lagrangian is

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 + f(t)x. \tag{4.32}$$

The answer is still of the form

$$\langle\, b \,|\, a \,\rangle = e^{\frac{i}{\hbar}S_{\rm cl}(b,\,a)}\, F(t_b - t_a). \tag{4.33}$$

To determine the new $S_{\rm cl}(b, a)$, we begin with the classical equation

$$m\ddot{x}_{\rm c}(\tau) + m\omega^2 x_{\rm c}(\tau) = f(\tau) \tag{4.34}$$

under the initial condition

$$x_{\rm c}(t_a) = x_a, \qquad x_{\rm c}(t_b) = x_b. \tag{4.35}$$

Solving (4.34) and (4.35), and substituting the result into the action integral, we have (see homework),

$$\begin{aligned} S_{\rm cl} = \frac{m\omega}{2\sin\omega(t_b - t_a)} \Bigg[& (x_a^2 + x_b^2)\cos\omega(t_b - t_a) - 2x_a x_b \\ & + \frac{2x_b}{m\omega}\int_{t_a}^{t_b} dt\, \sin\omega(t - t_a) f(t) \\ & + \frac{2x_a}{m\omega}\int_{t_a}^{t_b} dt\, \sin\omega(t_b - t) f(t) \\ & - \frac{2}{m^2\omega^2}\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2\, \sin\omega(t_b - t_1)\sin\omega(t_2 - t_a) f(t_1) f(t_2) \Bigg]. \end{aligned} \tag{4.36}$$

From $\langle\, b \,|\, a \,\rangle$, we can obtain the ground state to ground state transition amplitude $\langle\, 0 \,|\, 0 \,\rangle^f$. After a straightforward but somewhat tedious algebra, we obtain

$$\langle 0, t_b | 0, t_a \rangle^f = \langle 0, t_b | 0, t_a \rangle^{f=0}$$

$$\times \exp\left(-\frac{i}{2}\int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 f(t_1) D(t_1-t_2) f(t_2)\right) \tag{4.37}$$

where

$$D(t_1-t_2) \equiv \frac{-i}{2m\omega} e^{-i\omega|t_1-t_2|} \tag{4.38}$$

is a Green's function obeying

$$-m\left(\frac{d^2}{dt^2}+\omega^2\right) D(t-t') = \delta(t-t'). \tag{4.39}$$

3. Hamiltonian Path Integrals

In our derivation of path integrals, we have introduced a set of k-integrations and then carry out these k-integrations explicitly. We obtain the Hamiltonian path integrals by keeping these momentum integrations in the definition of path integrals.

In Eq. (4.6), we encounter

$$\langle x_2, t_2 | x_1, t_1 \rangle = \int \frac{dk}{2\pi} e^{-iH(t_2-t_1)+ik(x_2-x_1)}$$

$$= \int \frac{dk}{2\pi} e^{i(k\dot{x}-H)(t_2-t_1)}. \tag{4.40}$$

Hence, we have

$$\langle b | a \rangle = \int \frac{dk_1}{2\pi} dx_1 \frac{dk_2}{2\pi} \cdots dx_{n-1} \frac{dk_n}{2\pi} e^{i\int d\tau (k\dot{x}-H)}$$

$$= \int \mathscr{D}\{p, q\}\, e^{i\int d\tau (p\dot{q}-H(p,q))}. \tag{4.41}$$

In the final expression, we have replaced k and x by their generic names p and q. Note that in $\mathscr{D}\{p, q\}$, or $\mathscr{D}\{p\}\,\mathscr{D}\{q\}$, it no longer contains the normalization factors, $1/\sqrt{2\pi i\Delta t}$.

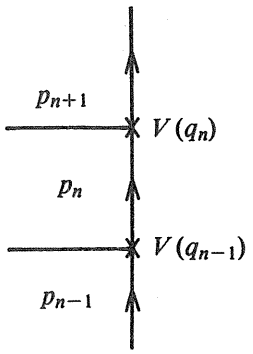


Fig. 4.2 The order of p, q integrations in the Hamiltonian path integrals.

We wish to point out that p_n, q_n, are not variables at the same time. Under our formulation, we can view the particle interacts with the potential $V(q)$ at the location q_n at time t_n. Between t_{n-1} and t_n, and between t_n and t_{n+1}, the particle travels with momenta P_n and P_{n+1}, as shown in Fig. 4.2.

The simplicity of (4.40) is very misleading. It suggests that if we know the classical Hamiltonian, we can determine the quantum-mechanical amplitude uniquely. We know that this conclusion is wrong because the quantum mechanics Hamiltonians such as

$$H_{qp} = q^2p^2 \tag{4.42}$$

$$H_{pq} = p^2q^2 \tag{4.43}$$

are different even though their classical limits are the same. Equations (4.42) and (4.43) are simple examples of different quantum mechanics orderings. Corresponding to (4.42) and (4.43), we may introduce

(a) qp-ordering: The operator q is always on the left of operator p in the Taylor's expansion of $H(p, q)$. E.g.

$$(p^2q + pq^2)_{qp} = qp^2 + q^2p.$$

(b) pq-ordering: The operator p is always on the left of operator q in the Taylor's expansion of $H(p, q)$.

There is another important ordering known as Weyl ordering which we shall not discuss it here.

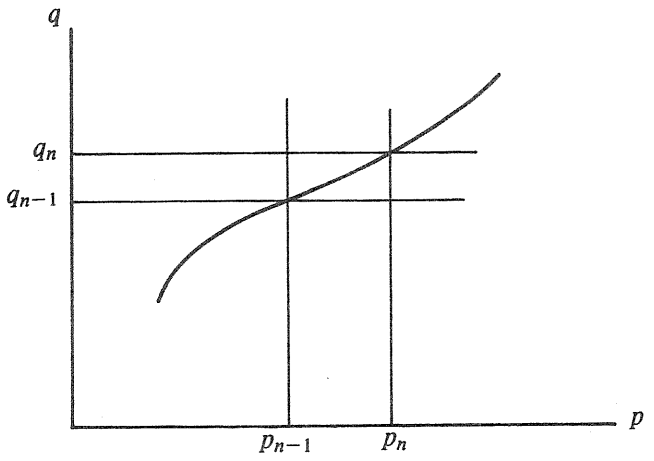


Fig. 4.3 A path in $p\,q$ space.

We wish to find the meaning of these orderings in the path integral formulation. To evaluate the integrand of (4.41), we divide the phase spaces into cells. We show a typical cell on the path in Fig. 4.3. We need to evaluate the matrix element associated with $\exp[i\Delta t\Sigma(pq - H(p, q)]$ on these cells. To obtain the numerical results of these two orderings, we need to approximate the exponents in two different ways:

(a) For the pq-ordering, we assign

$$i\Delta t\,\Sigma(p\dot{q} - H) \to i\,\Sigma\,[p_k(q_k - q_{k-1}) - \Delta t\,H(p_k, q_{k-1})].$$

(b) For the pq-ordering, we assign

$$i\Delta t\,\Sigma(p\dot{q} - H) \to i\,\Sigma\,[p_{k-1}(q_k - q_{k-1}) - \Delta t\,H(p_{k-1}, q_k)].$$

See F. A. Berezin [*Teor i Mat Fiz.* Vol 6, #2 pp194–212 (Feb, 1971) Translation pp141–154] for detailed discussions and for a similar result on Weyl ordering.

4. Scalar Field Theories and Feynman Rules

We consider a free scalar field ϕ coupled to an external current $j(x)$,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 + \phi(x)\,j(x). \tag{4.44}$$

In (4.44), we may include an interaction $V(\phi)$. Just as in the non-relativistic quantum mechanics system, we can use the path integral

method to describe a transition matrix element from t_a to t_b. Actually, we only need to know the vacuum to vacuum transition amiplitude in the presence of $j(x)$. We choose the physical ground state energy as zero, and consider the limit $t_b = \infty$, $t_a = -\infty$. We also choose the current such that $j = 0$ at $t \to \pm\infty$. Then we have

$$\langle 0|0\rangle^j = \int \mathscr{D}\{\phi(x)\}\, e^{i\int d^4x \mathscr{L}(\phi)} \tag{4.45}$$

where the free theory obeys the normalization

$$\langle 0|0\rangle^{j=0} = 1. \tag{4.46}$$

In the present case, we can calculate $\langle 0 \mid 0 \rangle^j$ exactly without doing any explicit calculation. We first obtain a classical solution in the presence of $j(x)$. We can derive the classical equation by

$$\delta \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_c)^2 - \frac{m^2}{2}\phi_c^2 + \phi_c j\right] = 0 \tag{4.47}$$

giving

$$-(\partial^2 + m^2)\,\phi_c(x) + j(x) = 0. \tag{4.48}$$

To invert (4.48), we need to know the boundary condition. We follow Feynman to obtain the causual solution

$$\phi_c(x) = -\int d^4x'\, \Delta_F(x-x')\, j(x'). \tag{4.49}$$

Then we define a new field ϕ' via

$$\phi(x) = \phi_c(x) + \phi'(x). \tag{4.50}$$

It is easy to see that $\phi'(x)$ is a free field by construction. We obtain

$$\langle 0|0\rangle^j = \int \mathscr{D}\{\phi'\} \exp\left\{ i \int d^4x \right.$$

$$\left. \times \left[\frac{1}{2}(\partial_\mu \phi_c + \partial_\mu \phi')^2 - \frac{m^2}{2}(\phi_c + \phi')^2 + (\phi_c + \phi')\, j\right]\right\}$$

$$= \int \mathscr{D}\{\phi'\} \exp\left\{ i \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_c)^2 - \frac{m^2}{2}\phi_c^2 + \phi_c j\right]\right.$$

$$+ \int d^4x \left[\partial_\mu \phi_c \, \partial_\mu \phi' - m^2 \phi_c \phi' + \phi' j\right]$$

$$\left. + \int d^4x \left[\frac{1}{2}(\partial_\mu \phi')^2 - \frac{m^2}{2}\phi'^2\right]\right\}. \tag{4.51}$$

The first term in the exponent of (4.51) gives the classical action, the second term vanishes due to (4.48), and the last term is indeed the free scalar Lagrange function in terms of ϕ'. Hence, we have

$$\langle 0|0\rangle^j = e^{iS_{cl}} \int \mathscr{D}\{\phi'\} \exp i \int d^4x \left[\frac{1}{2}(\partial_\mu \phi')^2 - \frac{m^2}{2}\phi'^2\right]$$

$$= e^{iS_{cl}} \langle 0|0\rangle^{j=0} \tag{4.52}$$

where

$$\begin{aligned} iS_{cl} &= i \int d^4x \left[\frac{1}{2}(\partial_\mu \phi_c)^2 - \frac{m^2}{2}\phi_c^2 + \phi_c j\right] \\ &= i \int d^4x \left[-\frac{1}{2}\phi_c (\partial^2 + m^2)\,\phi_c + \phi_c j\right] \\ &= \frac{i}{2} \int d^4x \, \phi_c(x)\, j(x) \\ &= -\frac{i}{2} \int d^4x \, d^4x' \, j(x)\, \Delta_F(x-x')\, j(x'), \end{aligned} \tag{4.53}$$

and

$$\int \mathscr{D}\{\phi'\} \, e^{i\int d^4x \left[\frac{1}{2}(\partial_\mu \phi')^2 - \frac{m^2}{2}\phi'^2\right]} = \langle 0|0\rangle^{j=0} = 1. \tag{4.54}$$

Hence, we have

$$W_0\{j\} \equiv \frac{\langle 0|0\rangle^j}{\langle 0|0\rangle^{j=0}} = \exp\left\{-\frac{i}{2}\int d^4x\, d^4x'\, j(x)\,\Delta_F(x-x')\,j(x')\right\}. \quad (4.55)$$

Equation (4.55) is the generalization of (4.37) to a scalar field theory. $W_0(j)$ is called the generating functional of the free scalar particle Green's functions. We can generate the n-point Green's function by

$$W_0\{j\} \equiv \frac{\int \mathscr{D}\{\phi\} \exp i \int d^4x\,(\mathscr{L}_0(\phi) + \phi j)}{\int \mathscr{D}\{\phi\} \exp i \int d^4x\,\mathscr{L}_0(\phi)}$$

$$= \sum \frac{i^n}{n!} \int dx_1 \cdots dx_n\, j(x_1)\, j(x_2) \cdots j(x_n)$$

$$\times\, G_n^0(x_1, x_2, \cdots x_n). \quad (4.56)$$

where $\mathscr{L}_0(\phi)$ is the free Lagrange function. We then have for the free particle Green's function

$$G_n^0(x_1, x_2, \cdots, x_n) = \frac{\int \mathscr{D}\{\phi\}\, e^{i\int d^4x \mathscr{L}_0(\phi)}\,(\phi(x_1)\cdots\phi(x_n))}{\int \mathscr{D}\{\phi\}\, e^{i\int d^4x \mathscr{L}_0(\phi)}}. \quad (4.57)$$

Comparing the expansions (4.55) and (4.56), we have

$$\begin{aligned} G_2^0(x_1, x_2) &= i\Delta_F(x_1-x_2), \\ G_4^0(x_1, x_2, x_3, x_4) &= i\Delta_F(x_1-x_2)\, i\Delta_F(x_3-x_4) \\ &\quad + i\Delta_F(x_1-x_3)\, i\Delta_F(x_2-x_4) + i\Delta_F(x_1-x_4)\, i\Delta_F(x_2-x_3), \\ &\quad \cdots . \end{aligned} \quad (4.58)$$

Equation (4.58) is the Wick's theorem in the path integral formulation. The equivalence between these two formulations is

$$\int \mathscr{D}\{\phi\}\, e^{i\int d^4x \mathscr{L}_0(\phi)} (\phi(x_1)\cdots\phi(x_n)) \leftrightarrow \langle 0|\, T(\phi(x_1)\cdots\phi(x_n))\,|0\rangle. \quad (4.59)$$

Thus we have

$$\int \mathscr{D}\{\phi\}\, e^{i\int d^4x \mathscr{L}_0(\phi)} (\phi(x_1)\,\phi(x_2)) = i\Delta_F(x_1-x_2) \equiv \overbrace{\phi(x_1)\,\phi(x_2)} \quad (4.60a)$$

$$\int \mathscr{D}\{\phi\}\, e^{i\int d^4x \mathscr{L}_0(\phi)} (\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4))$$

$$= \overbrace{\phi(x_1)\,\phi(x_2)}\,\overbrace{\phi(x_3)\,\phi(x_4)} + \overbrace{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)}$$

$$+ \overbrace{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)}, \cdots . \quad (4.60b)$$

We are now ready to study the interaction system. Consider the simplest ϕ^4 interaction

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \phi j$$

$$= \mathscr{L}_0 - \frac{\lambda}{4!}\phi^4 + \phi j. \quad (4.61)$$

Then the properly normalized vacuum to vacuum amplitude is

$$W\{j\} = \langle 0|\,0\rangle^j \,/\, \langle 0|\,0\rangle^{j=0}$$

$$= \sum \frac{i^n}{n!} \int dx_1\, dx_2 \cdots dx_n\, j(x_1)\cdots j(x_n)$$

$$\times\, G_n(x_1, x_2, \cdots, x_n) \quad (4.62)$$

where

$$\langle 0|\,0\rangle^j = \int \mathscr{D}\{\phi\}\, e^{i\int d^4x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + \phi j\right]}. \quad (4.63)$$

Hence, we have

$$G_n(x_1, \cdots, x_n)$$

$$= \frac{\int \mathscr{D}\{\phi\}\,(\phi(x_1)\cdots\phi(x_n))\, e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right)}}{\int \mathscr{D}\{\phi\}\, e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right)}}. \tag{4.64}$$

We can develop a perturbation calculation of G_n by expanding the $\frac{\lambda}{4!}\phi^4$.

In the numerator of (4.64), we encounter the expansion

$$\sum_N \int \mathscr{D}\{\phi\}\,\phi(x_1)\cdots\phi(x_n)\frac{1}{N!}\left(\int d^4y \frac{-i\lambda}{4!}\phi^4(y)\right)^N e^{i\int d^4x \mathscr{L}_0}.$$

The evaluation of this term gives rise to the usual Dyson-Wick's expansion. The only effect of the denominator is to cancel out all the vacuum diagram contribution in the numerator.

For instance, the two-point Green's function is

$$G_2(x_1, x_2) = \frac{\int \mathscr{D}\{\phi\}\,\phi(x_1)\,\phi(x_2)\, e^{i\int d^4x\left(\mathscr{L}_0(x) - \frac{\lambda}{4!}\phi^4(x)\right)}}{\int \mathscr{D}\{\phi\}\, e^{i\int d^4x\left(\mathscr{L}_0(x) - \frac{\lambda}{4!}\phi^4(x)\right)}}$$

$$= \frac{\int \mathscr{D}\{\phi\}\,\phi(x_1)\,\phi(x_2)\sum_N \frac{1}{N!}\left(-\frac{i\lambda}{4!}\int d^4y\,\phi^4(y)\right)^N e^{i\int d^4x \mathscr{L}_0(x)}}{\int \mathscr{D}\{\phi\}\sum_N \frac{1}{N!}\left(-\frac{i\lambda}{4!}\int d^4y\,\phi^4(y)\right)^N e^{i\int d^4x \mathscr{L}_0(x)}}. \tag{4.65}$$

The zeroth order term in λ gives

$$G_2^0(x_1, x_2) = \overbrace{\phi(x_1)\,\phi}(x_2) = \Delta_F(x_1 - x_2).$$

The first order term in λ gives

$$G_2^1(x_1, x_2) = -\frac{i\lambda}{2}\int d^4y\, \overbrace{\phi(x_1)\,\phi}(y)\,\overbrace{\phi(y)\,\phi}(x_2)\,\overbrace{\phi(y)\,\phi}(y)$$

which is represented by Fig. 4.4.

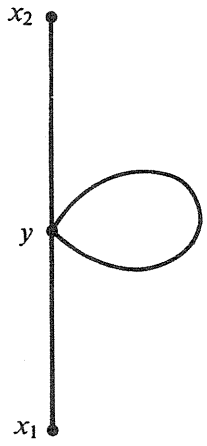


Fig. 4.4 The lowest order self-energy diagram.

The $O(\lambda)$ vacuum diagram described in Fig. 4.5,

$$-\frac{i\lambda}{8}\int dy\,(\overbrace{\phi(y)\,\phi}(y))^2\,\overbrace{\phi(x_1)\,\phi}(x_2),$$

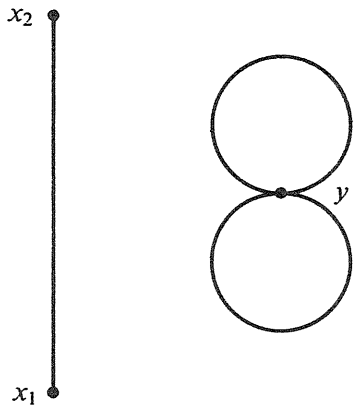


Fig. 4.5 A free 2-point Green's function with a vacuum diagram.

is cancelled by a corresponding term in the denominator.

We can generalize the above example to obtain a set of Feynman rules for computing the position-space Green's function. To compute $G_n(x_1, \ldots, x_n)$, we begin with n external vertices at $x_1, x_2, \ldots, x_n$. We then include an arbitrary number of internal vertices, depending on the order of the perturbation expansion. We then link these vertices such that there are 4 particle lines at each internal vertex. To compute the amplitude, we follow

(a) Draw all diagrams with n-external legs.
(b) For a contraction between two vertices at x and x', we associate with a factor $i\Delta_F(x-x')$

Fig. 4.6 Propagation function.

(c) For each internal vertex, we associate with a factor $-i\lambda/4!$.

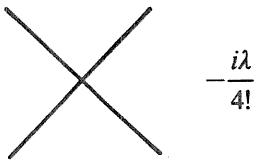


Fig. 4.7 Vertex amplitude.

The factor $1/4!$ usually cancels after we include all possible permutations of the contractions.

(d) We integrate over all internal coordinates, and then sum over the order N. To any order N, the overall factor $1/N!$ also usually cancels. There may be a symmetry factor $1/S$ remaining.
(e) We do not need to keep any vacuum diagram.

Knowing the position space Green's function, we can obtain the physical scattering amplitudes by assigning the on-shell momentum to all the external legs, and replacing these external propagators by the appropriate wave functions. Since the final Feynman rules are the same as those discussed in general field theory, we shall not repeat them here.

5. Dyson-Schwinger Equation

By using functional derivatives, we can put the Dyson-Wick expansion in a compact form. Consider Eq. (63),

$$\langle 0 \mid 0 \rangle^j = \int \mathscr{D}\{\phi\} \exp\left\{ i \int d^4x \mathscr{L}_0(\phi) - \frac{\lambda}{4!}\phi^4 + \phi j \right\}$$

$$= \int \mathscr{D}\{\phi\}\, e^{-\frac{i\lambda}{4!}\int d^4y\, \phi^4(y)} \exp\left\{ i \int d^4x \mathscr{L}_0(\phi) + \phi j \right\}. \quad (4.66)$$

Operating on the last factor, we can replace $\phi^4(y)$ by $\left(\frac{1}{i}\frac{\delta}{\delta j(y)}\right)^4$,

$$\langle 0|0\rangle^j = \int \mathscr{D}\{\phi\}\, e^{-\frac{i\lambda}{4!}\int d^4y \left(\frac{\delta}{i\delta j(y)}\right)^4} \exp\left\{i\int d^4x \mathscr{L}_0(\phi) + \phi j\right\}.$$

$$= e^{-\frac{i\lambda}{4!}\int d^4y \left(\frac{\delta}{i\delta j(y)}\right)^4} \int \mathscr{D}\{\phi\} \exp\left[i\int d^4x \mathscr{L}_0(\phi) + \phi j\right].$$

$$= e^{-\frac{i\lambda}{4!}\int d^4y \left(\frac{\delta}{i\delta j(y)}\right)^4} \exp\left\{-\frac{i}{2}\int d^4x_1\, d^4x_2\, j(x_1)\, \Delta_F(x_1-x_2)\, j(x_2)\right\}. \tag{4.67}$$

In (4.67), we have chosen the normalization such that the non interacting vacuum state obeys $\langle 0|0\rangle^{j=0} = 1$.

We reproduce the Dyson-Wick expansion by carrying out the functional differentiation explicitly.

For a general scalar Lagrange function

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - V(\phi) + \phi(x)\, j(x), \tag{4.68}$$

we obtain an analogous equation

$$\langle 0|0\rangle^j = e^{-i\int d^4y\, V\left(\frac{\delta}{i\delta j(y)}\right)}$$

$$\times \exp\left[-\frac{i}{2}\int d^4x_1\, d^4x_2\, j(x_1)\, \Delta_F(x_1-x_2)\, j(x_2)\right]. \tag{4.69}$$

Multiplying the exponential factor of $V\left(\frac{\delta}{i\delta j}\right)$ to the left, we have

$$e^{i\int d^4y\, V\left(\frac{\delta}{i\delta j(y)}\right)} \langle 0|0\rangle^j = W_0\{j\}$$

$$= \exp\left[-\frac{i}{2}\int d^4x_1\, d^4x_2\, j(x_1)\, \Delta_F(x_1-x_2)\, j(x_2)\right]. \tag{4.70}$$

We can verify easily that $W_0\{j\}$ obeys

$$(\partial^2+m^2)\frac{\delta}{i\delta j(x)}W_0\{j\}$$

$$=(\partial^2+m^2)\int d^4x'\,(-\Delta_F(x-x')\,j(x'))\,W_0\{j\}$$

$$=j(x)\,W_0\{j\}, \tag{4.71}$$

$$\left[(\partial^2+m^2)\frac{\delta}{i\delta j(x)}-j(x)\right]W_0\{j\}$$

$$=\left[(\partial^2+m^2)\frac{\delta}{i\delta j(x)}-j(x)\right]e^{i\int d^4y\,V\left(\frac{\delta}{i\delta j(y)}\right)}\langle 0|0\rangle^j$$

$$=0. \tag{4.72}$$

Moving the exponential factor in (4.72) through the factor on the left, we obtain

$$\left[(\partial^2+m^2)\frac{\delta}{i\delta j(x)}+V'\left(\frac{1}{i}\frac{\delta}{\delta j(x)}\right)-j(x)\right]W\{j\}=0, \tag{4.73}$$

where

$$W\{j\}\equiv\langle 0|0\rangle^j/\langle 0|0\rangle^{j=0} \tag{4.74}$$

is the generating functional for the interacting Green's functions. Equation (4.73) gives a complete set of functional differential equations relating different Green's functions, and is an example of Dyson-Schwinger equations. We may use (4.73) as the starting point of the quantum field theory calculation.

We wish to point out that we use the notation $\langle 0|0\rangle^{j=0}$ to describe the vacuum to vacuum amplitudes for both the free and the interaction theory. In reality, the two amplitudes differ by an infinite phase which reflects the energy difference between the interacting and the free vacuum states. This extra infinite phase corresponds to the sum over all disconnected vacuum diagrams, and is required to cancel the vacuum

diagrams in (4.74). Equation (4.73) is equivalent to the operator field equation

$$(\partial^2+m^2)\,\phi(x) + V'(\phi(x)) - j(x) = 0.$$

6. Grassmann Algebra

To define a path integral over the fermion system, we need to introduce Grassmann algebra. We consider a Grassmann algebra generated by n generators $\theta_1, \theta_2, \cdots \theta_n$ obeying

$$\theta_i\theta_j + \theta_j\theta_i \equiv \{\theta_i, \theta_j\} = 0. \tag{4.75}$$

Equation (4.75) implies, in particular, $\theta_i^2 = 0$ for all i. We denote a Grassmann algebra with n elements by $\mathcal{G}_n$. A convenient basis in $\mathcal{G}_n$ is the monomials

$$\begin{aligned} &1 \\ &\theta_1, \theta_2, \cdots, \theta_n \\ &\theta_1\,\theta_2, \theta_1\,\theta_3, \cdots, \theta_{n-1}\theta_n \\ &\cdots \\ &\theta_1\,\theta_2 \cdots \theta_n. \end{aligned} \tag{4.76}$$

Every element $F(\theta)$ of the algebra $\mathcal{G}_n$ can be represented by a linear combination of these monomials

$$F(\theta) = F^{(0)} + \sum_i F_i^{(1)}\theta_i + \sum F_{ij}^{(2)}\,\theta_i\theta_j$$

$$+ \cdots + \sum F_{ij\cdots k}^{(n)}\,\theta_i\theta_j \cdots \theta_k, \tag{4.77}$$

where the coefficients are totally antisymmetric in their indices. We can introduce sum, product, etc of these elements, $F(\theta) + G(\theta)$, $F(\theta)\,G(\theta)$. We can also introduce more complicated expressions such as

$$\exp(F(\theta)) \equiv 1 + F(\theta) + \frac{1}{2!}(F(\theta))^2 + \cdots. \tag{4.78}$$

We can express them easily in the form (4.77).

An element which contains only monomials of even degree in θ is called even. An element which contains only monomials of odd degree in θ is called odd. Every element of $\mathcal{G}_n$ can be expressed uniquely as the sum of an even and an odd element. It is easy to show that all even elements are commuting among themselves, and are also commuting with all odd elements. All odd elements are anticommuting among themselves.

We introduce the differention $\dfrac{\partial}{\partial\theta_i}$ as follows:

$$\frac{\partial}{\partial\theta_i}(\theta_1\theta_2\cdots\theta_m) = \delta_{i1}\,\theta_2\theta_3\cdots\theta_m - \delta_{i2}\,\theta_1\theta_3\cdots\theta_m$$

$$+ \cdots + (-1)^{m-1}\,\delta_{im}\,\theta_1\theta_2\cdots\theta_{m-1}. \tag{4.79}$$

We introduce $\dfrac{\overleftarrow{\partial}}{\partial\theta_i}$ as

$$(\theta_1\theta_2\cdots\theta_m)\frac{\overleftarrow{\partial}}{\partial\theta_i} = \delta_{mi}\,\theta_1\theta_2\cdots\theta_{m-1} + \cdots$$

$$+ (-1)^{m-1}\,\delta_{1i}\,\theta_2\theta_3\cdots\theta_m. \tag{4.80}$$

It is straightforward to show that

$$\left\{\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\theta_j}\right\} \equiv \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} = 0, \tag{4.81}$$

$$\left\{\frac{\partial}{\partial\theta_i}, \theta_j\right\} = \delta_{ij}. \tag{4.82}$$

Thus, we may interpret θ_i and $\dfrac{\partial}{\partial\theta_i}$ as the fermion annihilation and creation operators.

In addition to the differentiation, we can introduce integrations over Grassmann algebra. We define

$$\int d\theta_i = 0 \tag{4.83a}$$

$$\int d\theta_i \, \theta_j = \delta_{ij} \tag{4.83b}$$

where $d\theta_i$ obeys

$$\{d\theta_i, d\theta_j\} = 0 = \{\theta_i, d\theta_j\}. \tag{4.84}$$

We can introduce multiple integrals such as

$$\int d\theta_n \, d\theta_{n-1} \cdots d\theta_1 \, F(\theta) = n! \, F^{(n)}_{12\cdots n}. \tag{4.85}$$

The integration and differentiation give the same result.

We can show that the integration by part formulae on the Grassmann algebra are valid

$$\int d\theta_n \cdots d\theta_1 \, F(\theta) \frac{\partial}{\partial \theta_i} G(\theta)$$

$$= \int d\theta_n \cdots d\theta_1 \left(F(\theta) \frac{\overleftarrow{\partial}}{\partial \theta_i} \right) G(\theta). \tag{4.86}$$

Another important formula is the Gaussian integral for an even number of generators,

$$\int d\theta_n \cdots d\theta_1 \exp\left(\frac{1}{2} \theta_i A_{ij} \theta_j \right) = \sqrt{\det A} \tag{4.87}$$

with $A_{ij} = -A_{ji}$ being an antisymmetric matrix. We shall work out the $n=2$ case here

For $n=2$, we have

$$\exp\left(\frac{1}{2} \theta_i A_{ij} \theta_j \right) = \exp\left(A_{12} \theta_1 \theta_2 \right)$$

$$= 1 + A_{12} \theta_1 \theta_2 .$$

Hence, we have

$$\int d\theta_2\, d\theta_1 \exp\left(\frac{1}{2}\theta_i A_{ij}\theta_j\right)$$

$$= \int d\theta_2\, d\theta_1 (1 + A_{12}\theta_1\theta_2)$$

$$= A_{12} = \sqrt{\det A}$$

where the determinant of A is

$$\det A = \begin{vmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{vmatrix} = (A_{12})^2.$$

It is known that the determinant of an even rank antisymmetric matrix is always a perfect square. (The determinant of an odd rank antisymmetric matrix is zero.) The square root of this determinant is known as the Pfaffian of the matrix. For $n = 2$, we have

$$\text{Pfaffian} = A_{12} = \frac{1}{2}\varepsilon_{ij} A_{ij}.$$

For $n = 4$, we have

$$\text{Pfaffian} = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23}$$

$$= \frac{1}{4}\varepsilon_{ijkl} A_{ij} A_{kl}.$$

We can generalize easily the above result to an arbitrary n.

7. Fermion System

We are now ready to introduce a path integral over a Dirac field

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\psi}\eta + \bar{\eta}\psi \tag{4.88}$$

where η and $\bar{\eta}$ are anticommuting external currents, obeying

$$\{\eta, \eta\} = \{\eta, \bar{\eta}\} = \{\bar{\eta}, \bar{\eta}\}$$

$$= \{\psi, \eta\} = \{\bar{\psi}, \eta\} = \{\psi, \bar{\eta}\} = \{\bar{\psi}, \bar{\eta}\} = 0. \tag{4.89}$$

We can write down the vacuum to vacuum transition amplitude as

$$\langle 0|0\rangle^{\eta,\bar{\eta}} = \int \mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\} \times \exp\left\{i\int d^4x\,[\bar{\psi}(i\partial\!\!\!/ - m)\,\psi + \bar{\psi}\eta + \bar{\eta}\psi]\right\}. \quad (4.90)$$

Following the bose system, we introduce an analogous "classical solution" by requiring

$$\delta\int d^4x\,[\bar{\psi}_c\,(i\partial\!\!\!/ - m)\,\psi_c + \bar{\psi}_c\eta + \bar{\eta}\,\psi_c] = 0$$

and obtaining

$$(i\partial\!\!\!/ - m)\,\psi_c + \eta = 0 \quad (4.91a)$$

$$\bar{\psi}_c\,(-i\overleftarrow{\partial\!\!\!/} - m) + \bar{\eta} = 0. \quad (4.91b)$$

As fermion fields, ψ_c and $\bar{\psi}_c$ are anticommuting among themselves and with respect to η, $\bar{\eta}$. From (4.91), we have

$$\psi_c(x) = -\int d^4x'\,S_F(x-x')\,\eta(x') \quad (4.92a)$$

$$\bar{\psi}_c(x) = -\int d^4x'\,\bar{\eta}(x')\,S_F(x'-x). \quad (4.92b)$$

Expanding ψ and $\bar{\psi}$ around ψ_c and $\bar{\psi}_c$, we have

$$\psi(x) = \psi_c(x) + \psi'(x) \quad (4.93a)$$

$$\bar{\psi}(x) = \bar{\psi}_c(x) + \bar{\psi}'(x). \quad (4.93b)$$

Substituting (4.93) into (4.90), we can show that the linear terms in $\mathscr{L}$ drop out, giving

$$\langle 0|0\rangle^{\eta,\bar{\eta}} = \int \mathscr{D}\{\psi'\}\,\mathscr{D}\{\bar{\psi}'\}$$

$$\times \exp\left\{ i\int d^4x\,[\psi_c(i\partial\!\!\!/ - m)\,\psi_c + \bar{\psi}_c\eta + \bar{\eta}\psi + \bar{\psi}'(i\partial\!\!\!/ - m)\,\psi']\right\}$$

$$= e^{iS_{cl}}\langle 0|0\rangle^{\eta,\bar{\eta}=0} \tag{4.94}$$

where

$$iS_{cl} = i\int d^4x\,[\bar{\psi}_c\,(i\partial\!\!\!/ - m)\,\psi_c + \bar{\psi}_c\eta + \bar{\eta}\psi_c]$$

$$= -i\int d^4x\,d^4x'\,\bar{\eta}(x)\,S_F\,(x-x')\,\eta(x'). \tag{4.95}$$

Hence, we have the generating functional for the free Dirac fields as

$$W_0\{\eta,\bar{\eta}\} \equiv \langle 0|0\rangle^{\eta,\bar{\eta}} / \langle 0|0\rangle^{\eta,\bar{\eta}=0}$$

$$= \exp\left[-i\int d^4x\,d^4x'\,\bar{\eta}(x)\,S_F\,(x-x')\,\eta(x')\right]. \tag{4.96}$$

By expanding around the external currents η, $\bar{\eta}$, we reproduce the Dyson-Wick expansion for the fermion fields.

We can consider a Lagrange function in the presence of both the scalar and the Dirac fields,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\,\psi$$

$$+ \phi j + \bar{\psi}\eta + \bar{\eta}\psi. \tag{4.97}$$

It is straightforward to show that the generating functional for the free scalar and free Dirac fields is

$$W_0\{j,\eta,\bar{\eta}\} \equiv \frac{\langle 0|0\rangle^{j,\eta,\bar{\eta}}}{\langle 0|0\rangle^{j\cdot\eta\cdot\bar{\eta}=0}}$$

$$= \exp\left\{-\frac{i}{2}\int d^4x\,d^4x'\,j(x)\,\Delta_F\,(x-x')\,j(x')\right.$$

$$- i \int d^4x\, d^4x'\, \bar{\eta}(x))\, S_F(x-x')\, \eta(x') \Big\}. \tag{4.98}$$

Knowing $W_0\{j, \eta, \bar{\eta}\}$, we are ready to evaluate the Green's functions of an interacting theory such as

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + \bar{\psi}(i\partial\!\!\!/ - m)\,\psi$$

$$- V(\phi) - ig\bar{\psi}\gamma_5\psi\phi + \phi j + \bar{\psi}\eta + \bar{\eta}\psi. \tag{4.99}$$

We can express the generating functional of Green's functions of the full theory as

$$W\{j, \eta, \bar{\eta}\} \equiv \frac{\langle 0|0\rangle^{j,\eta,\bar{\eta}}}{\langle 0|0\rangle^{j,\eta,\bar{\eta}=0}}$$

$$= \frac{\int \mathscr{D}\{\phi\}\,\mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\} \exp i \int d^4x\,[\mathscr{L}_0 - V(\phi) - ig\bar{\psi}\gamma_5\psi\phi + \phi j + \bar{\psi}\eta + \bar{\eta}\psi]}{\langle 0|0\rangle^{j,\eta,\bar{\eta}=0}}$$

$$= \exp i \int d^4x \left[-V\left(\frac{\delta}{i\delta j(x)}\right) \right.$$

$$\left. -ig\left(-\frac{\delta}{i\delta\eta(x)}\gamma_5\frac{\delta}{i\delta\bar{\eta}(x)}\right)\frac{\delta}{ij(x)}\right] W_0\{j, \eta, \bar{\eta}\}. \tag{4.100}$$

Evaluating the functional derivatives order by order, we obtain the required Feynman rules.

Note that $W_0\{j, \eta, \bar{\eta}\}$ obeys

$$\left[-(\partial^2 + m^2)\frac{\delta}{i\delta j(x)} + j(x)\right] W_0\{j, \eta, \bar{\eta}\} = 0 \tag{4.101}$$

$$\left[(i\partial\!\!\!/ - m)\frac{\delta}{i\delta\bar{\eta}(x)} + \eta(x)\right] W_0\{j, \eta, \bar{\eta}\} = 0, \tag{4.102}$$

and an adjoint equation of (4.102). From these equations, we can derive the Dyson-Schwinger equations as

$$\left[(\partial^2 + m^2)\frac{\delta}{i\delta j(x)} + V'\left(\frac{1}{i}\frac{\delta}{\delta j(x)}\right)\right.$$

$$\left. + ig\left(-\frac{\delta}{i\delta\eta(x)}\gamma_5\frac{\delta}{i\delta\bar{\eta}(x)}\right) - j(x)\right] W\{j, \eta, \bar{\eta}\} = 0 \tag{4.103}$$

$$\left[(i\partial\!\!\!/ - m)\frac{\delta}{i\delta\bar{\eta}(x)} - ig\,\gamma_5\frac{\delta}{i\delta\bar{\eta}(x)}\frac{\delta}{i\delta j(x)} + \eta(x)\right] W\{j, \eta, \bar{\eta}\} = 0. \tag{4.104}$$

As we have mentioned earlier, we can take these equations as the starting point of the quantum field theory calculation. Equations (4.104) are equivalent to the operator field equations

$$(\partial^2 + m^2)\,\phi(x) + V'(\phi(x)) + ig\,\bar{\psi}(x)\,\gamma_5\,\psi(x) - j(x) = 0,$$

$$(i\partial\!\!\!/ - m)\,\psi(x) - ig\,\gamma_5\,\psi(x)\,\phi(x) + \eta(x) = 0.$$

We can use these analogies to write down Dyson-Schwinger equations readily.

Chapter 5

GAUGE THEORIES

In this Chapter, we use the path integral method to derive the Feynman rules for gauge theories. We begin with the familiar case of Maxwell field which is an Abelian gauge theory. We then generalize our result to Non-Abelian gauge theories.

1. Path Integral Formulation of Maxwell Fields

The Lagrange function for a Maxwell field is

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - A_\mu j_\mu \tag{5.1}$$

with

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{5.2}$$

When we apply the path integral method to the Maxwell field naively, we encounter the expression

$$\langle 0 | 0 \rangle^j = \int \mathscr{D}\{A\}\, e^{i \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - A_\mu j_\mu \right]} . \tag{5.3}$$

According to our standard method, we first obtain the classical field equation

$$\partial_\mu F^{\mathrm{C}}_{\mu\nu} = \partial_\mu \left(\partial_\mu A^{\mathrm{C}}_\nu - \partial_\nu A^{\mathrm{C}}_\mu \right) = j_\nu . \tag{5.4}$$

Unfortunately, Eq. (5.4) does not have an inverse. From (5.4), we can only determine A^{C}_μ up to a gradient term. This lack of inverse is a consequence of gauge invariance: Eq. (5.4) is invariant under the gauge transformation,

$$A_\mu \to A_\mu + \partial_\mu \lambda . \tag{5.5}$$

Another important property of the gauge field is that the current j_μ coupled to the gauge field must be a conserved current. We can then show that the action

$$S \equiv \int d^4x \, \mathscr{L} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - A_\mu j_\mu \right) \tag{5.6}$$

is invariant under the gauge transformation (5.5). When we separate $\mathscr{D}(A)$ into a gauge-dependent non-gradient part $\mathscr{D}^{\mathrm{T}}(A)$ and a gradient part $\mathscr{D}(\lambda)$, we have

$$\int \mathscr{D}\{A\} \, e^{i\int d^4x \, \mathscr{L}(A)} = \int \mathscr{D}\{\lambda\} \int \mathscr{D}^{\mathrm{T}}\{A\} \, e^{i\int d^4x \, \mathscr{L}(A)} . \tag{5.7}$$

Since the $\mathscr{D}^{\mathrm{T}}(A)$ integral is λ-independent due to the gauge invariant, we encounter a divergent integral $\int \mathscr{D}\{\lambda\}$. This is a trivial divergence, and describes the total "gauge group volume". We shall factor out this divergent "group volume" factor from the definition of $\langle 0 | 0 \rangle^j$, and restrict our integration over the finite hyperspace $\{A^{\mathrm{T}}\}$. Different methods of separating the group volume lead to different gauge conditions. We shall work out a few cases explicitly.

(a) Coulomb gauge

We first decompose A_k into

$$A_k = A'_k + \partial_k \lambda \tag{5.8}$$

with

$$\nabla \cdot \mathbf{A}' = 0. \tag{5.9}$$

In terms of $\mathbf{A}'$, we have

$$\mathscr{D}\{A\} = \mathscr{D}\{A'\} \tag{5.10}$$

$$S \equiv \int d^4x \, \mathscr{L}(A) = \int d^4x \, \mathscr{L}(A'). \tag{5.11}$$

To factor out the gauge condition, we make use of the identity

$$\int \mathscr{D}\{\lambda\} \, (\det \nabla^2) \, \delta\{\nabla \cdot \mathbf{A} - \nabla^2 \lambda\} = 1. \tag{5.12}$$

Then, we find

$$\langle 0|0\rangle^j = \int \mathscr{D}\{A\} \int \mathscr{D}\{\lambda\} (\det \nabla^2)\, \delta\{\nabla\cdot\mathbf{A}-\nabla^2\lambda\}\, e^{i\int d^4x \mathscr{L}(A)}$$

$$= \int \mathscr{D}\{\lambda\} (\det \nabla^2) \int \mathscr{D}\{A'\}\, \delta\{\nabla\cdot\mathbf{A}'\}\, e^{i\int d^4x \mathscr{L}(A')}$$

$$= (\text{gauge group volume}) \int \mathscr{D}\{A'\}\, \delta\{\nabla\cdot\mathbf{A}'\}\, e^{i\int d^4x \mathscr{L}(A')}. \tag{5.13}$$

As we have mentioned earlier, we should divide out the gauge group volume factor and define a finite vacuum to vacuum transition amplitude

$$\langle 0|0\rangle'^j \equiv \int \mathscr{D}\{A\}\, \delta\{\nabla\cdot\mathbf{A}\}\, e^{i\int d^4x \left\{-\frac{1}{4}F_{\mu\nu}F_{\mu\nu} - A_\mu j_\mu\right\}}, \tag{5.14}$$

with the normalization

$$\langle 0|0\rangle'^{j=0} = 1. \tag{5.15}$$

We may use our previous method to evaluate $\langle 0|0\rangle'^j$ explicitly. We can obtain the classical field equation in the Coulomb gauge as

$$-\partial_k F^{\mathrm{C}}_{k0} - j_0 = 0 \tag{5.16a}$$

$$\partial_\mu F^{\mathrm{C}}_{\mu k} - j_k + \partial_k \Lambda = 0. \tag{5.16b}$$

Note that in the Coulomb gauge, $\nabla\cdot\delta\mathbf{A}=0$, we cannot use the variational principle to determine the longitudinal part of (5.16b). We include the gradient term as a Lagrange multiplier, and determine it by requiring $\nabla\cdot\mathbf{A}^{\mathrm{C}} = 0$. From (5.16a), we obtain the Coulomb potential,

$$A^{\mathrm{C}}_0(x) = \int d^3x' \frac{1}{4\pi|x-x'|} j_0(x'). \tag{5.17}$$

From (5.16b), we have

$$\partial^2 A_k^{\mathrm{C}} = j_k + \partial_\mu \partial_k A_\mu^{\mathrm{C}} - \partial_k \Lambda$$

$$= j_k - \partial_k \Lambda'. \tag{5.18}$$

The Coulomb gauge condition implies

$$-\nabla^2 \Lambda' + \partial_k j_k = 0$$

or

$$\Lambda' = (\nabla^2)^{-1} \partial_k j_k \tag{5.19}$$

where $(\nabla^2)^{-1}$ is a shorthand of the Coulomb Green's function $-1/4\pi \,|x-x'|$. Substituting (5.19) into (5.18), we have

$$\partial^2 A_k^{\mathrm{C}} = j_k - (\nabla^2)^{-1} \partial_k \partial_l j_l \equiv \Lambda_{kl} j_l \tag{5.20}$$

and consequently,

$$A_k^{\mathrm{C}}(x) = (\partial^2)^{-1} \Lambda_{kl} j_l = \int d^4x' \, D_{\mathrm{F}}(x-x') \, \Lambda_{kl} j_l(x') \tag{5.21}$$

where $D_{\mathrm{F}}(x-x')$ is the massless propagation function obeying

$$\partial^2 D_{\mathrm{F}}(x-x') = \delta^4(x-x'). \tag{5.22}$$

We can decompose

$$A_\mu = A_\mu^{\mathrm{C}} + A'_\mu \tag{5.23}$$

where

$$\nabla \cdot \mathbf{A}^{\mathrm{C}} = \nabla \cdot \mathbf{A}' = 0. \tag{5.24}$$

Then, we have

$$\langle 0|0\rangle'^{\,j} = e^{iS_{\mathrm{cl}}} \int \mathscr{D}\{A'\} \, \delta\{\nabla \cdot \mathbf{A}'\} \, e^{i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F_{\mu\nu}\right)}$$

$$= e^{iS_{\mathrm{cl}}} \langle 0|0\rangle'^{\,j=0} \tag{5.25}$$

or

$$\frac{\langle 0|0\rangle^{j}}{\langle 0|0\rangle^{j=0}} = e^{iS_{\mathrm{cl}}}$$

$$= \exp\left\{\frac{i}{2}\int d^4x\,[-A_0^{\mathrm{C}} j_0 + A_k^{\mathrm{C}} j_k]\right\}$$

$$= \exp\left\{\frac{i}{2}\int d^4x \int d^4x' \left[j_0(x)\frac{\delta(x_0-x_0')}{4\pi|x-x'|} j_0(x')\right.\right.$$

$$\left.\left. + j_k(x)\,(-D_{\mathrm{F}}(x-x'))\,\Lambda_{kl} j_l(x')\right]\right\}. \tag{5.26}$$

From (5.26), we can obtain the propagator and the Dyson-Wick theorem for the Coulomb gauge Maxwell field.

(b) Lorentz gauge

With a little modification, we can apply the path integral method to the Lorentz gauge,

$$\partial_\mu A_\mu = 0. \tag{5.27}$$

To factor out the gauge group volume, we use the identity

$$\int \mathscr{D}\{\lambda\}\,\det\partial^2\,\delta\,[\partial_\mu A_\mu - \partial^2\lambda] = 1. \tag{5.28}$$

We decompose an arbitrary gauge field by

$$A_\mu = A'_\mu + \partial_\mu\lambda \tag{5.29}$$

with

$$\partial_\mu A'_\mu = 0. \tag{5.30}$$

We then have

$$\langle 0|0\rangle^{j} = \int \mathscr{D}\{A\}\, e^{i\int d^4x\,\mathscr{L}(A)}$$

$$= \int \mathscr{D}\{A'\} \int \mathscr{D}\{\lambda\}\,\det\partial^2\,\delta\{\partial_\mu A_\mu - \partial^2\lambda\}\, e^{i\int d^4x\,\mathscr{L}(A')}$$

$$= \int \mathscr{D}\{\lambda\} \det \partial^2 \int \mathscr{D}\{A'\}\, \delta\{\partial_\mu A'_\mu\}\, e^{i\int d^4x \mathscr{L}(A')}$$

$$= (\text{gauge group volume}) \int \mathscr{D}\{A\}\, \delta\{\partial_\mu A_\mu\}\, e^{i\int d^4x \mathscr{L}(A)}. \tag{5.31}$$

Dropping out the group volume, we define a finite vacuum transition amplitude

$$\langle 0|0\rangle^{\prime\, j} = \int \mathscr{D}\{A\}\, \delta\{\partial_\mu A_\mu\}\, e^{i\int d^4x \mathscr{L}(A)}. \tag{5.32}$$

We can prove without any difficulty that

$$\frac{\langle 0|0\rangle^{\prime\, j}}{\langle 0|0\rangle^{\prime\, j=0}} = e^{iS_{\text{cl}}}$$

$$= \exp\left\{-\frac{i}{2}\int d^4x d^4x'\, j_\mu(x) \right.$$

$$\left. \times \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\right)(-D_{\text{F}}(x-x'))j_\nu(x')\right\}. \tag{5.33}$$

From (5.33), we can identify the Lorentz gauge propagation function as

$$-i\,(D_{\text{F}\mu\nu}(x-x'))_{\text{Lorentz gauge}} = -i\left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}\right) D_{\text{F}}(x-x'). \tag{5.34}$$

(c) Gaussian Gauge

There are other methods of extracting out the infinite group volume. For instance, instead of δ-functionals, we may insert a Gaussian damping factor. Using the identity

$$\int \mathscr{D}\{\lambda\}\, e^{-\frac{i}{2\beta}\int d^4x (\partial_\mu A_\mu - \partial^2\lambda)^2} (\text{determinant factor}) = 1, \tag{5.35}$$

we may introduce, after dropping out the group volume, a finite vacuum to vacuum transition amplitude

$$\langle 0|0\rangle'^{j} = \int \mathscr{D}\{A\}\, e^{-\frac{i}{2\beta}\int d^4x\,(\partial_\mu A_\mu)^2}$$

$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - A_\mu j_\mu \right] \right\}$$

$$= \int \mathscr{D}\{A\} \exp\left\{ i \int d^4x \left[\frac{1}{2} A_\mu (\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu) A_\nu \right.\right.$$

$$\left.\left. - \frac{1}{2\beta}(\partial_\mu A_\mu)^2 - A_\mu j_\mu \right] \right\}. \tag{5.36}$$

As long as $\beta^{-1} \neq 0$, the new field equation is invertible. The classical field equation reads

$$(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu) A_\nu^{\mathrm{C}} + \frac{1}{\beta} \partial_\mu \partial_\nu A_\nu^{\mathrm{C}} - j_\mu = 0. \tag{5.37}$$

Differentiating (5.37) with respect to μ, we have

$$\frac{1}{\beta} \partial^2 (\partial_\nu A_\nu^{\mathrm{C}}) - \partial_\mu j_\mu = 0$$

or

$$\partial_\nu A_\nu^{\mathrm{C}} = \beta (\partial^2)^{-1} \partial_\mu j_\mu . \tag{5.38}$$

Substituting (5.38) into (5.37), we have

$$\partial^2 A_\mu^{\mathrm{C}} = j_\mu + \left(1 - \frac{1}{\beta}\right) \partial_\mu (\partial_\nu A_\nu^{\mathrm{C}})$$

$$= \left[g_{\mu\nu} - (1-\beta) \frac{\partial_\mu \partial_\nu}{\partial^2} \right] j_\nu . \tag{5.39}$$

We obtain, as before,

$$\frac{\langle 0|0\rangle'^{j}}{\langle 0|0\rangle'^{j=0}} = e^{iS_{\mathrm{cl}}}$$

$$= \exp\left[-\frac{i}{2}\int d^4x\, A^{\mathrm{C}}_{\mu} j_{\mu}\right]$$

$$= \exp\left[-\frac{i}{2}\int d^4x\, d^4x'\, j_{\mu}(x)\right.$$

$$\left.\times\left(g_{\mu\nu} - (1-\beta)\frac{\partial_{\mu}\partial_{\nu}}{\partial^2}\right)(-D_{\mathrm{F}}(x-x'))j_{\nu}(x')\right]. \quad (5.40)$$

The propagation function in this gaue is

$$-iD_{\mathrm{F}\mu\nu}(x-x') = -i\left(g_{\mu\nu} - (1-\beta)\frac{\partial_{\mu}\partial_{\nu}}{\partial^2}\right)D_{\mathrm{F}}(x-x'). \quad (5.41)$$

We recover the Lorentz gauge condition by choosing $\beta=0$. Indeed, at $\beta=0$, the Gaussian reduces to a δ-funcitonal. At $\beta=1$, we obtain the Feynman gauge propagator $-ig_{\mu\nu}D_{\mathrm{F}}(x-x')$.

2. Quantum Electrodynamics

Consider the Lagrange function for the quantum electrodynamics (QED)

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}\left[i(\partial\!\!\!/ - ieA\!\!\!/) - m\right]\psi. \quad (5.42)$$

The Lagrange function is invariant under a coordinate dependent phase transformation

$$\psi(x) \to \psi'(x) = e^{-ie\lambda(x)}\psi(x) \quad (5.43a)$$

$$\psi^{\dagger}(x) \to \psi'^{\dagger}(x) = \psi^{\dagger}(x)\, e^{ie\lambda(x)} \quad (5.43b)$$

provided that $A_{\mu}(x)$ transforms according to

$$A_{\mu}(x) \to A'_{\mu}(x) = A_{\mu}(x) - \partial_{\mu}\lambda(x). \quad (5.44)$$

Even though $\partial_{\mu}\psi$ does not transform simply, the combination $(\partial_{\mu} - ieA_{\mu})\psi(x)$ does:

$$(\partial_\mu - ieA_\mu(x))\,\psi(x) \to (\partial_\mu - ieA'_\mu(x))\,\psi'(x)$$

$$= (\partial_\mu - ie(A_\mu(x) - \partial_\mu \lambda(x))\, e^{-ie\lambda(x)}\,\psi(x)$$

$$= e^{-ie\lambda(x)}\,(\partial_\mu - ieA_\mu(x))\,\psi(x). \tag{5.45}$$

Then, it is simple to see that

$$\mathscr{L}'(x) = \mathscr{L}(x).$$

We call the combined transformation (5.43) and (5.44) the gauge transformation. The QED Lagrange function is invariant under such a transformation.

We are ready to introduce the path integral formulation of QED. We consider the Lagrange function in the presence of external currents

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \bar{\psi}\,[i(\partial\!\!\!/ - ieA\!\!\!/) - m]\,\psi$$

$$- A_\mu j_\mu + \bar{\eta}\psi + \bar{\psi}\eta. \tag{5.46}$$

It is easy to see that the formal path integral leads to a divergent group volume factor. We need to introduce a cutoff in the gauge parameter space. A convenient expression for the vacuum to vacuum transition amplitude is

$$\langle 0 \mid 0 \rangle^{j,\eta,\bar{\eta}} = \int \mathscr{D}\{A\}\,\mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\}\, e^{-\frac{i}{2\beta}\int d^4x (\partial_\mu A_\mu)^2}$$

$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(i(\partial\!\!\!/ - ieA\!\!\!/) - m)\psi \right.\right.$$

$$\left.\left. - A_\mu j_\mu + \bar{\eta}\psi + \bar{\psi}\eta \right]\right\}$$

$$= \exp\left\{ i \int d^4x \left[e\left(-\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)\gamma_\mu \left(-\frac{1}{i}\frac{\delta}{\delta j_\mu(x)}\right)\frac{\delta}{i\delta\bar{\eta}(x)} \right]\right\}$$

$$\times \int \mathscr{D}\{A\}\,\mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\}\, e^{-\frac{i}{2\beta}\int d^4x(\partial_\mu A_\mu)^2}$$

$$\times \exp\left\{ i\int d^4x\left[-\frac{1}{4}F_{\mu\nu}^2+\bar{\psi}(i\partial\!\!\!/-m)\psi-A_\mu j_\mu+\bar{\eta}\psi+\bar{\psi}\eta \right]\right\}. \tag{5.47}$$

Hence, the generating functional for the QED Green functions obeys

$$W\{j,\eta,\bar{\eta}\} \equiv \frac{\langle 0|0\rangle^{j,\eta,\bar{\eta}}}{\langle 0|0\rangle^{j,\eta,\bar{\eta}=0}}$$

$$= \exp\left\{ i\int d^4x\, e\left(-\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)\gamma_\mu\left(-\frac{1}{i}\frac{\delta}{\delta j_\mu(x)}\right)\frac{\delta}{i\delta\bar{\eta}(x)}\right\}$$

$$\times\, W_0\{j,\eta,\bar{\eta}\} \tag{5.48}$$

where

$$W_0\{j,\eta,\bar{\eta}\} = \exp\left[-\frac{i}{2}\int d^4x\, d^4x'\, j_\mu(x)\right.$$

$$\times (g_{\mu\nu}-(1-\beta)\,\partial_\mu\partial_\nu/\partial^2)\,(-D_F(x-x'))\,j_\nu(x')$$

$$\left. -\, i\int d^4x\, d^4x'\,\bar{\eta}(x)\,S_F(x-x')\,\eta(x')\right]. \tag{5.49}$$

As we have mentioned earlier, Eq. (5.48) is not really correct because the left hand side of (5.48) contains the vacuum diagram contribution. This leads to an infinite phase factor. However, if we evaluate (5.48)–(5.49) by perturbation theory and ignore all disconnected vacuum diagrams, then, we recover the correct answer. Setting $\beta = 1$ and evaluating (5.48) and (5.49) term by term, we reproduce the Feynman rules for QED as described in the general field theory.

The Dyson-Schwinger equations for QED are

$$\left[\left(i\partial\!\!\!/ - m + e\,\gamma_\mu\left(-\frac{1}{i}\frac{\delta}{\delta j_\mu(x)}\right)\right)\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)} + \eta(x)\right] W\{j,\eta,\bar{\eta}\} = 0 \tag{5.50a}$$

$$\left[\left(\partial^2 g_{\mu\nu} - \left(1-\frac{1}{\beta}\right)\partial_\mu\partial_\nu\right)\left(-\frac{1}{i}\frac{\delta}{\delta j_\nu(x)}\right)\right.$$

$$\left. + e\left(-\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)\gamma_\mu\left(\frac{\delta}{i\delta\bar{\eta}(x)}\right) - j_\mu(x)\right] W\{j,\eta,\bar{\eta}\} = 0. \qquad (5.50b)$$

We may use these equations to relate Green's functions of different orders.

3. Yang-Mills Fields

A Yang-Mills field is the simplest example of Non-Abelian gauge theories. Yang and Mills wished to extend the gauge transformation in QED to include the isospin, and asked the following question: Under what condition is a theory invariant under a coordinate and isospin dependent phase transformation?

A coordinate-dependent isospin phase transformation is

$$\psi'(x) = e^{-ig\frac{\tau}{2}\cdot\omega(x)}\psi(x) \qquad (5.51)$$

where ψ is both a spinor and an isospinor, and τ is the Pauli matrix. In analogy to QED, it is natural to introduce three gauge fields $\mathbf{B}_\mu = (B^1_\mu, B^2_\mu, B^3_\mu)$ and require that the interaction on ψ is through the combination $(\partial_\mu - ig\,(\tau/2)\cdot\mathbf{B}_\mu)\,\psi(x)$. The coefficient g is the coupling strength. We can determine the transformation law of $\mathbf{B}_\mu$ by requiring that the above combination transforms covariantly,

$$\left(\partial_\mu - ig\frac{\tau}{2}\cdot\mathbf{B}_\mu(x)\right)\psi(x) \to \left(\partial_\mu - ig\frac{\tau}{2}\cdot\mathbf{B}'_\mu(x)\right)\psi'(x)$$

$$= \left(\partial_\mu - ig\frac{\tau}{2}\cdot\mathbf{B}'_\mu(x)\right)e^{-ig\frac{\tau}{2}\cdot\omega(x)}\psi(x)$$

$$= e^{-ig\frac{\tau}{2}\cdot\omega(x)}\left(\partial_\mu - ig\frac{\tau}{2}\cdot\mathbf{B}_\mu(x)\right)\psi(x). \qquad (5.52)$$

Hence, we have

$$\partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}'_\mu(x) = e^{-ig\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\omega}(x)}\left(\partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}_\mu(x)\right)e^{ig\frac{\boldsymbol{\tau}}{2}\cdot\boldsymbol{\omega}(x)}. \tag{5.53}$$

This is a rather messy expression. However, the expression for an infinitesimal $\boldsymbol{\omega}$ is both simple and informative. Replacing $\boldsymbol{\omega}$ by $\delta\boldsymbol{\omega}$ in (5.53) and keeping terms linear in $\delta\boldsymbol{\omega}$, we have

$$\begin{aligned}\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}'_\mu(x) &= \left(1 - ig\frac{\boldsymbol{\tau}}{2}\cdot\delta\boldsymbol{\omega}\right)\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}_\mu(x)\left(1 + ig\frac{\boldsymbol{\tau}}{2}\cdot\delta\boldsymbol{\omega}\right)\\ &\quad - \frac{\boldsymbol{\tau}}{2}\cdot\partial_\mu\,\delta\boldsymbol{\omega} + O((\delta\omega)^2)\\ &= \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}_\mu(x) - ig\left[\frac{\boldsymbol{\tau}}{2}\cdot\delta\boldsymbol{\omega}, \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}_\mu(x)\right] - \frac{\boldsymbol{\tau}}{2}\cdot\partial_\mu\,\delta\boldsymbol{\omega} + O((\delta\omega)^2)\\ &= \frac{\boldsymbol{\tau}}{2}\cdot[\mathbf{B}_\mu(x) + g\,\delta\boldsymbol{\omega}\times\mathbf{B}_\mu(x) - \partial_\mu\,\delta\boldsymbol{\omega}] + O((\delta\omega)^2).\end{aligned} \tag{5.54}$$

Thus, we obtain

$$\begin{aligned}\delta\,\mathbf{B}_\mu &\equiv \mathbf{B}'_\mu(x) - \mathbf{B}_\mu(x)\\ &= -\partial_\mu\,\delta\boldsymbol{\omega} + g\,\delta\boldsymbol{\omega}\times\mathbf{B}_\mu + O((\delta\omega)^2).\end{aligned} \tag{5.55}$$

Equation (5.55) reveals that $\mathbf{B}_\mu$ transforms both as gauge field ($-\partial_\mu\,\delta\boldsymbol{\omega}$), and as an isovector ($g\delta\boldsymbol{\omega}\times\mathbf{B}_\mu$).

Knowing the transformation law of $\mathbf{B}_\mu$, we can construct a set of tensor fields $\mathbf{G}_{\mu\nu}$ which transform covariantly under the gauge transformation without the additional gradient terms.

It is easy to see that the simple combination $\partial_\mu\,\mathbf{B}_\nu - \partial_\nu\,\mathbf{B}_\mu$ does not have the required transformation property,

$$\begin{aligned}\delta(\partial_\mu\,\mathbf{B}_\nu - \partial_\nu\,\mathbf{B}_\mu) &= \partial_\mu\,\delta\mathbf{B}_\nu - \partial_\nu\,\delta\mathbf{B}_\mu\\ &= g\,\delta\boldsymbol{\omega}\times(\partial_\mu\,\mathbf{B}_\nu - \partial_\nu\,\mathbf{B}_\mu)\\ &\quad + g\,(\partial_\mu\,\delta\boldsymbol{\omega}\times\mathbf{B}_\nu - \partial_\nu\,\delta\boldsymbol{\omega}\times\mathbf{B}_\mu).\end{aligned} \tag{5.56}$$

With the help of (5.55), we can eliminate the $\partial_\mu \delta\boldsymbol{\omega}$ terms in (5.56), and obtain

$$\delta(\partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu) = g\,\delta\boldsymbol{\omega} \times (\partial_\mu \mathbf{B}_\nu - \partial_\nu \delta\mathbf{B}_\mu)$$

$$+ g\,[(-\delta\mathbf{B}_\mu + g\,\delta\boldsymbol{\omega} \times \mathbf{B}_\mu) \times \mathbf{B}_\nu - (-\delta\mathbf{B}_\nu + g\,\delta\boldsymbol{\omega} \times \mathbf{B}_\nu) \times \mathbf{B}_\mu]$$

$$= -g\,\delta(\mathbf{B}_\mu \times \mathbf{B}_\nu) + g\,\delta\boldsymbol{\omega} \times (\partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu + g\,\mathbf{B}_\mu \times \mathbf{B}_\nu). \tag{5.57}$$

Thus, we find that the combination,

$$\mathbf{G}_{\mu\nu} \equiv \partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu + g\,\mathbf{B}_\mu \times \mathbf{B}_\nu. \tag{5.58}$$

transforms covariantly under the gauge transformation,

$$\delta\,\mathbf{G}_{\mu\nu} \equiv \mathbf{G}'_{\mu\nu} - \mathbf{G}_{\mu\nu} = g\,\delta\boldsymbol{\omega} \times \mathbf{G}_{\mu\nu}\,. \tag{5.59}$$

The tensor field $\mathbf{G}_{\mu\nu}$ is the analog of $F_{\mu\nu}$.

A covariant Lagrange function which is invariant under the local isospin rotation is

$$\mathscr{L} = -\frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu} + \bar{\psi}\left[i\gamma_\mu\left(i\partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}_\mu\right) - m\right]\psi \tag{5.60}$$

with

$$\mathbf{G}_{\mu\nu} \equiv \partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu + g\,\mathbf{B}_\mu \times \mathbf{B}_\nu. \tag{5.61}$$

We can use the action principle to quantize the Yang-Mills field in the transverse gauge, and then verify the Lorentz invariance of the theory. We shall not enter into the details here. In the following, we shall concentrate on the path integral formulation of the Yang-Mills field.

It is sometimes more convenient to use the matrix notation to describe the Yang-Mills fields and their field equations. (See homework). Let $\boldsymbol{\tau}$ be 2×2 Pauli matrices. We introduce the matrix notation

$$G_{\mu\nu} \equiv \mathbf{G}_{\mu\nu}\cdot\frac{\boldsymbol{\tau}}{2}, \tag{5.62}$$

$$B_\mu \equiv \mathbf{B}_\mu\cdot\frac{\boldsymbol{\tau}}{2}, \tag{5.63}$$

$$D_\mu \equiv \partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot \mathbf{B}_\mu. \tag{5.64}$$

We can express (5.60) and (5.61) in the matrix form as

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}\,(G_{\mu\nu}\,G_{\mu\nu}) + \bar{\psi}(i\,\gamma_\mu\,D_\mu - m)\psi, \tag{5.65}$$

$$G_{\mu\nu} \equiv \partial_\mu\,B_\nu - \partial_\nu\,B_\mu + \frac{g}{i}[B_\mu, B_\nu]$$

$$= \frac{i}{g}[D_\mu, D_\nu]. \tag{5.66}$$

The field equations obtained from (5.60) are

$$\partial_\mu\,\mathbf{G}_{\mu\nu} + g\,\mathbf{B}_\mu \times \mathbf{G}_{\mu\nu} + g\,\bar{\psi}\,\gamma_\nu\frac{\boldsymbol{\tau}}{2}\psi = 0 \tag{5.67}$$

$$\left[i\gamma_\mu\left(\partial_\mu - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}_\mu\right) - m\right]\psi = 0. \tag{5.68}$$

We can express (5.67) and (5.68) in the matrix form as

$$[D_\mu, G_{\mu\nu}] + j_\nu = 0 \tag{5.69}$$

$$(i\gamma_\mu\,D_\mu - m)\,\psi = 0 \tag{5.70}$$

and

$$j_\nu \equiv \frac{\boldsymbol{\tau}}{2}\cdot\mathbf{j}_\nu, \qquad \mathbf{j}_\nu \equiv g\,\bar{\psi}\,\gamma_\nu\frac{\boldsymbol{\tau}}{2}\psi. \tag{5.71}$$

In analog to the identity in QED,

$$\partial_\lambda\,F_{\mu\nu} + \partial_\mu\,F_{\nu\lambda} + \partial_\nu\,F_{\lambda\mu} = 0, \tag{5.72}$$

we have the Bianchi identities,

$$[D_\lambda, G_{\mu\nu}] + [D_\mu, G_{\nu\lambda}] + [D_\nu, G_{\lambda\mu}] = 0. \tag{5.73}$$

Using the Jacobi identity and (5.66), we can show that

$$[D_\mu, [D_\nu, G_{\mu\nu}]]$$

$$= \frac{1}{2}\{[D_\mu, [D_\nu, G_{\mu\nu}]] - [D_\nu, [D_\mu, G_{\mu\nu}]]\}$$

$$= \frac{1}{2}[[D_\mu, D_\nu], G_{\mu\nu}] = 0. \tag{5.74}$$

This implies that the current j_μ coupled to the gauge field obeys

$$[D_\mu, j_\mu] = 0, \tag{5.75a}$$

or, in the isovector notation,

$$\partial_\mu \mathbf{j}_\mu + g\,\mathbf{B}_\mu \times \mathbf{j}_\mu = 0. \tag{5.75b}$$

Equation (5.75) is the analog of the current conservation in the Yang-Mills theory.

4. Path Integrals and Feynman Rules

We shall use the path integral method to derive the Feynman rules of the Yang-Mills theory. In the following, we shall concentrate on the Lorentz gauge. We can generalize the results to other gauges as well.

Consider an infinitesimal gauge transformation

$$\mathbf{B}'_\mu = \mathbf{B}_\mu + g\,\delta\boldsymbol{\omega} \times \mathbf{B}_\mu - \partial_\mu \delta\boldsymbol{\omega} + O((\delta\omega)^2). \tag{5.76}$$

Then, we have

$$\partial_\mu \mathbf{B}'_\mu = \partial_\mu \mathbf{B}_\mu + g\,\partial_\mu\,\delta\boldsymbol{\omega} \times \mathbf{B}_\mu + g\,\delta\boldsymbol{\omega} \times \partial_\mu \mathbf{B}_\mu$$

$$- \partial^2\,\delta\boldsymbol{\omega} + O((\delta\omega)^2). \tag{5.77}$$

The Lorentz condition $\partial_\mu \mathbf{B}'_\mu = 0$ implies, to $O((\delta\omega)^2)$,

$$\partial_\mu \mathbf{B}_\mu + g\,\partial_\mu\,\delta\boldsymbol{\omega} \times \mathbf{B}_\mu - \partial^2\,\delta\boldsymbol{\omega} = 0. \tag{5.78}$$

We construct the identity

$$\int \mathscr{D}\{\delta\boldsymbol{\omega}\} \det A\, \delta\{\partial_\mu B^a_\mu - A^{ab}\,\delta\omega^b\} = 1 \tag{5.79}$$

where

$$A^{ab} \equiv \partial^2\,\delta^{ab} - g\,\varepsilon^{abc}\,B^c_\mu\,\partial_\mu, \tag{5.80}$$

$$\det A = \det \partial^2 \det\left(\delta^{ab} - g\varepsilon^{abc}\,(\partial^2)^{-1}\,B^c_\mu\,\partial_\mu\right). \tag{5.81}$$

We consider a Yang-Mills field in the presence of an external current $\mathbf{j}_\mu$.

$$\mathscr{L} = -\frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \mathbf{B}_\mu\cdot\mathbf{j}_\mu. \tag{5.82}$$

Consistency requires that $\mathbf{j}_\mu$ obeys (5.75).

The naive vacuum to vacuum transition amplitude contains an infinite gauge group volume factor,

$$\begin{aligned}\langle 0|0\rangle^j &= \int \mathscr{D}\{\mathbf{B}_\mu\} \int \mathscr{D}\{\delta\boldsymbol{\omega}\} \det\{A\} \\ &\quad \times \delta\{\partial_\mu \mathbf{B}_\mu + g\,\partial_\mu\,\delta\boldsymbol{\omega}_\mu \times \mathbf{B}_\mu - \partial^2\delta\boldsymbol{\omega}\}\, e^{i\int d^4x\,\mathscr{L}(\mathbf{B}_\mu)} \\ &= \int \mathscr{D}\{\delta\boldsymbol{\omega}\} \det\partial^2 \int \mathscr{D}\{\mathbf{B}_\mu\}\,\delta\{\partial_\mu\,\mathbf{B}_\mu\} \\ &\quad \times \det\left\{1 - g\,(\partial^2)^{-1}\,B_\mu\,\partial_\mu\right\} e^{i\int d^4x\,\mathscr{L}(B_\mu)}.\end{aligned} \tag{5.83}$$

Dropping the group volume, we have the finite transition matrix in the Lorentz gauge,

$$\begin{aligned}\langle 0|0\rangle'^j &= \int \mathscr{D}\{\mathbf{B}_\mu\}\,\delta\{\partial^\mu\,\mathbf{B}_\mu\} \det\left\{1 - g\,(\partial^2)^{-1}\,\mathbf{B}_\mu\,\partial_\mu\right\} \\ &\quad \times e^{i\int d^4x\left(-\frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \mathbf{B}_\mu\cdot\mathbf{j}_\mu\right)}.\end{aligned} \tag{5.84}$$

It is known that for any matrix M

$$\ln \det M = \mathrm{Tr} \ln M$$

which we can verify easily by diagnoizing M. Thus, we have

$$\det \left\{1 - g\,(\partial^2)^{-1}\, \mathrm{B}_\mu\, \partial_\mu\right\} = e^{\mathrm{Tr}\ln\left\{1 - g\,(\partial^2)^{-1}\, B_\mu \partial_\mu\right\}}, \tag{5.85}$$

and consequently,

$$\langle 0|0\rangle^{\prime\, j} = \int \mathscr{D}\{\mathbf{B}_\mu\}\, \delta\{\partial_\mu \mathbf{B}_\mu\}\, e^{\mathrm{Tr}\ln\left\{1 - g\,(\partial^2)^{-1}\, B_\mu \partial_\mu\right\}}$$

$$\times\, e^{i\int d^4x \left[-\frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \mathbf{B}_\mu \cdot \mathbf{j}_\mu\right]}. \tag{5.86}$$

We can use (5.86) as the starting point of the perturbation calculation. However, the Tr ln term is difficult to handle.

Feynman invented a trick to take care of this Tr ln term. Let ϕ be a complex (i.e. charged) scalar field. We have by integration

$$\int \mathscr{D}\{\phi\}\, \mathscr{D}\{\phi^*\}\, e^{i\int d^4x \left[\phi^{a*}\left(\partial^2 \delta^{ab} - g\,\varepsilon^{abc} B_\mu^c\, \partial_\mu\right)\phi^b\right]}$$

$$= \frac{1}{\det A}$$

$$= e^{-\mathrm{Tr}\ln\left(1 - g\,(\partial^2)^{-1}\, B_\mu \partial_\mu\right)} / \det \partial^2. \tag{5.87}$$

The det ∂^2 has no **B**-dependence, and can be absorbed into the normalization. If we change Tr to (-1)Tr, then (5.87) will reduce to (5.58). Feynman said that if ϕ is a scalar fermion, then we have to change Tr to (-1)Tr, as required by all fermion loops. If we accept this interpretation, we can rewrite (5.84) as

$$\langle 0|0\rangle^{\prime j} = \int \mathscr{D}\{\mathbf{B}_\mu\}\, \delta\{\partial_\mu \mathbf{B}_\mu\} \int \mathscr{D}\{\phi\}\, \mathscr{D}\{\phi^*\}$$

$$\times \exp\left\{i\int d^4x \left[-\frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu}\right.\right.$$

$$+ \phi^{a*} (\partial^2 \delta^{ab} - g\varepsilon^{abc} B_\mu^c \partial_\mu) \phi^b - \mathbf{B}_\mu \cdot \mathbf{j}_\mu \Big] \Big\}. \tag{5.88}$$

Thus, we can obtain the Feynman rules from the following effective Lagrange function with ghost fields ϕ and ϕ^*,

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4} \mathbf{G}_{\mu\nu} \cdot \mathbf{G}_{\mu\nu}$$

$$+ \phi^{a*} (\partial^2 \delta^{ab} - g\, \varepsilon^{abc} B_\mu^c \partial_\mu) \phi^b - \mathbf{B}_\mu \cdot \mathbf{j}_\mu. \tag{5.89}$$

Under the gauge condition, we have

$$\partial_\mu \mathbf{B}_\mu = 0. \tag{5.90}$$

It is important to remember that ϕ is a fictitious particle introduced for the evaluation of det A. It does not correspond to any physical particle. Hence, the introduction of a scalar fermion here does not violate the Spin-Statistics relation. We refer ϕ as a massless scalar ghost field. (Faddeev-Povpov ghost.)

The generalization of the above result to a gaussian gauge is less transparent. We shall outline the derivation proposed by 't Hooft (*Nuclear Physics* **B33** (1971) 173) without getting into the details. Consider a family of gauge conditions

$$\partial_\mu B_\mu^a (x) - C^a (x) = 0. \tag{5.91}$$

The average of a *gauge-invariant* function $X(\mathbf{B})$ with respect to the gauge condition (5.91) is

$$\langle X \rangle = \frac{\int \mathscr{D}\{\mathbf{B}\} \det (A\{\mathbf{B}\})\, \delta\{\partial_\mu \mathbf{B}_\mu - \mathbf{C}\}\, X\{\mathbf{B}\}\, e^{i\int \mathscr{L} d^4x}}{\int \mathscr{D}\{\mathbf{B}\} \det (A\{\mathbf{B}\})\, \delta\{\partial_\mu \mathbf{B}_\mu - \mathbf{C}\}\, e^{i\int \mathscr{L} d^4x}}. \tag{5.92}$$

Both the numerator and the denominator of (5.92) are independent of $\mathbf{C}(x)$. Thus we can weighted-average the numerator and the denominator of $\mathbf{C}(x)$ without affecting the ratio,

$$\langle X \rangle = \frac{\int \mathscr{D}\{\mathbf{C}\}\, e^{-\frac{i}{2\beta}\int d^4x\, \mathbf{C}(x)^2} \int \mathscr{D}\{\mathbf{B}\}\, \det A\, \delta\{\partial_\mu \mathbf{B}_\mu - \mathbf{C}\}\, X\{\mathbf{B}\}\, e^{i\int \mathscr{L} d^4x}}{\int \mathscr{D}\{\mathbf{C}\}\, e^{-\frac{i}{2\beta}\int d^4x\, \mathbf{C}(x)^2} \int \mathscr{D}\{\mathbf{B}\}\, \det A\, \delta\{\partial_\mu \mathbf{B}_\mu - \mathbf{C}\}\, e^{i\int d^4x \mathscr{L}}}$$

$$= \frac{\int \mathscr{D}\{\mathbf{B}\}\, e^{-\frac{i}{2\beta}\int d^4x (\partial_\mu \mathbf{B}_\mu)^2} \det A\{\mathbf{B}\}\, X\{\mathbf{B}\}\, e^{i\int \mathscr{L} d^4x \mathscr{L}}}{\int \mathscr{D}\{\mathbf{B}\}\, e^{-\frac{i}{2\beta}\int d^4x (\partial_\mu \mathbf{B}_\mu)^2} \det A\{\mathbf{B}\}\, e^{i\int d^4x \mathscr{L}}}. \tag{5.93}$$

Expression (5.93) is exactly what we would expect from a gaussian gauge. We can also introduce ghost fields ϕ and ϕ^* to take care of $\det A\{\mathbf{B}\}$. Thus, for the purpose of computing gauge invarianat quantities, we can start with

$$\langle 0|0\rangle' = \int \mathscr{D}\{\mathbf{B}\}\, \mathscr{D}\{\phi\}\, \mathscr{D}\{\phi^*\}\, e^{-\frac{i}{2\beta}\int d^4x (\partial_\mu \mathbf{B}_\mu)^2}$$

$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu} + \phi^{a*}(\partial^2\delta^{ab} - g\varepsilon^{abc}B_\mu^c\partial_\mu)\phi^b - \mathbf{B}_\mu\cdot\mathbf{j}_\mu \right] \right\}. \tag{5.94}$$

We can include a $(\partial_\mu B_\mu^a)^2$ term in the effective Lagrange function, giving

$$\mathscr{L}_{\text{eff}} = -\frac{1}{4}\mathbf{G}_{\mu\nu}\cdot\mathbf{G}_{\mu\nu} - \frac{1}{2\beta}(\partial_\mu \mathbf{B}_\mu)^2 + \phi^{a*}(\partial^2\delta^{ab} - g\varepsilon^{abc}B_\mu^c\partial_\mu)\phi^b$$

$$- \mathbf{B}_\mu\cdot\mathbf{j}_\mu + \mathscr{L}_{\text{fermion}}. \tag{5.95}$$

In (5.95), we have also included the contribution due to a fermion field ψ,

$$\mathscr{L}_{\text{fermion}} = \bar{\psi}\left[i\left(\partial\!\!\!/ - ig\frac{\boldsymbol{\tau}}{2}\cdot\mathbf{B}\!\!\!/ \right) - m \right]\psi$$

$$+ \bar{\psi}\eta + \bar{\eta}\psi. \tag{5.96}$$

The gauge-fixing term is included explicitly in (5.95). We can separate the effective Lagrange function $\mathscr{L}_{\text{eff}}$ into a quadratic "free" part $\mathscr{L}_0$ and a

higher-order "interaction" part $\mathscr{L}_{\text{int}}$,

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_0 + \mathscr{L}_{\text{int}} \tag{5.97}$$

with

$$\mathscr{L}_0 = -\frac{1}{4}(\partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu)^2 - \frac{1}{2\beta}(\partial_\mu \mathbf{B}_\mu)^2 + \phi^{a*}\partial^2\delta^{ab}\phi^b$$

$$+ \bar{\psi}\, i(\not{\partial} - m)\psi - \mathbf{B}_\mu \cdot \mathbf{j}_\mu + \bar{\psi}\eta + \bar{\eta}\psi, \tag{5.98a}$$

$$\mathscr{L}_{\text{int}} = -\frac{g}{2}\varepsilon^{abc}(\partial_\mu B_\nu^a - \partial_\nu B_\mu^a) B_\mu^b B_\nu^c - \frac{1}{4} g^2 \varepsilon^{abf}\varepsilon^{cdf} B_\mu^a B_\nu^b B_\mu^c B_\nu^d$$

$$- g\,\varepsilon^{abc}\phi^{a*}\partial_\mu \phi^b B_\mu^c + g\,\bar{\psi}\frac{\tau^a}{2}\gamma_\mu \psi B_\mu^a. \tag{5.98b}$$

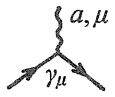
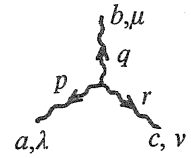
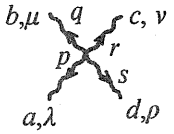
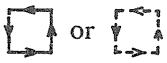
From $\mathscr{L}_0$, we obtain the usual gaussian integration and the propagators. From $\mathscr{L}_{\text{int}}$, we obtain the effective interaction among the fields and the generating functional for the Green's functions

$$W\{j, \eta, \bar{\eta}\} \equiv \frac{\langle 0|0\rangle^{j,\eta,\bar{\eta}}}{\langle 0|0\rangle^{j,\eta,\bar{\eta}=0}}. \tag{5.99}$$

We can derive the Feynman rules in a straightforward way. We summarize the Feynman rules for Yang-Mills fields coupled to a Dirac field in Table V.

Table V Feynman rules for Yang-Mills fields coupled to a Dirac field in the gaussian gauge. The Lorentz gauge corresponds to $\beta = 0$.

Processes	Feynman Rules	Remark
p	$\dfrac{i}{\not{p} - m + i\varepsilon}$	Dirac Propagator
k; b,ν; a,μ	$\dfrac{-i\delta^{ab}\left(g_{\mu\nu} - \dfrac{(1-\beta)\,k_\mu k_\nu}{k^2}\right)}{k^2 + i\varepsilon}$	Yang-Mills Propagator in Gaussian Gauge
q; a; b	$\delta^{ab}\dfrac{-i}{q^2 + i\varepsilon}$	Ghost Propagator

	$ig\,\gamma_\mu \dfrac{\tau^a}{2}$	
	$-g\,\varepsilon^{abc}[(p-q)_\nu g_{\lambda\mu}$ $+\ (q-r)_\lambda g_{\mu\nu} + (r-p)_\mu g_{\nu\lambda}\}$	Arrows here indicate the directions of momentum
	$-ig^2\varepsilon^{abf}\varepsilon^{cdf}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\rho}g_{\mu\nu})$ $-ig^2\varepsilon^{acf}\varepsilon^{bdf}(g_{\lambda\mu}g_{\nu\rho}-g_{\lambda\rho}g_{\mu\nu})$ $-ig^2\varepsilon^{adf}\varepsilon^{cbf}(g_{\lambda\nu}g_{\mu\rho}-g_{\lambda\mu}g_{\rho\nu})$	f is summed over Arrows indicate momentum flows
	$-g\,\varepsilon^{abc}p_\lambda$ $\left[\text{or } -\dfrac{g}{2}\varepsilon^{abc}(p_\lambda+q_\lambda)\right]$	Momemtum p is associated with outgoing ghost
	-1	closed fermion or ghost loop

5. Examples

We shall use the Feynman rules to write down some second order amplitudes.

(a) Second-order self-energy contributions to gauge particles

There are four graphs contributing to the amplitude. See Fig. 5.1.

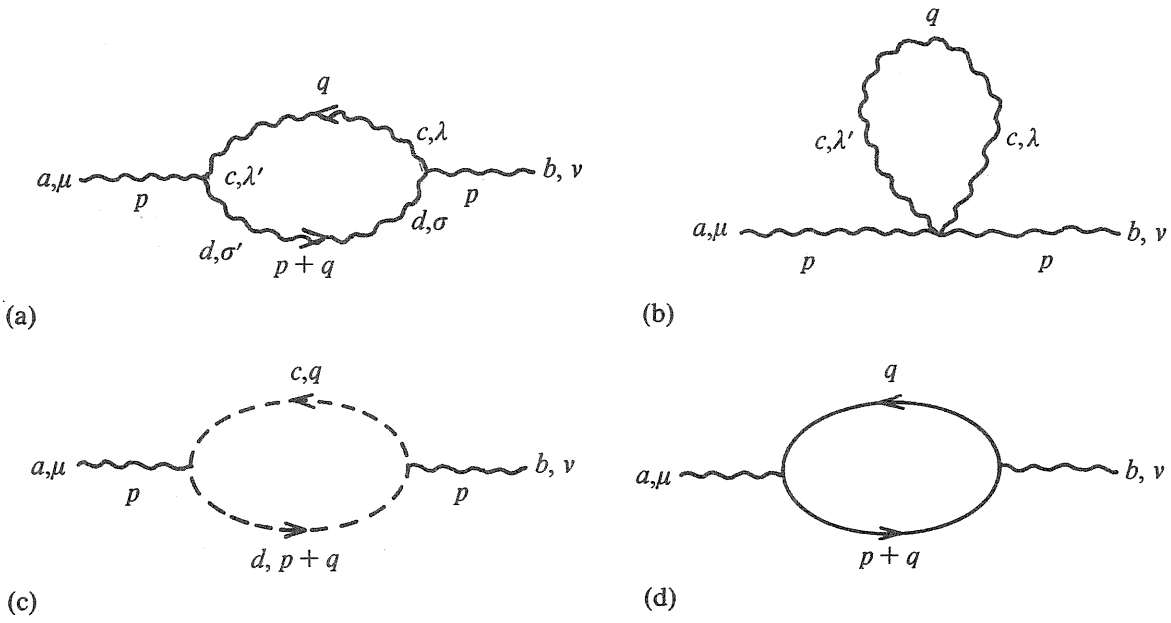


Fig. 5.1 Second-order self-energy contribution to gauge fields.

The amplitude for Fig. 5.1a is

$$\Pi^{(a)\,ab}_{\mu\nu}(p) = \frac{1}{2}\varepsilon^{bcd}\varepsilon^{acd}g^2\int\frac{d^4q}{(2\pi)^4}$$

$$\times[(p-q)_\sigma g_{\lambda\nu} + (2q+p)_\nu g_{\lambda\sigma} + (-2p-q)_\lambda g_{\sigma\nu}]$$

$$\times[(p-q)_{\sigma'} g_{\lambda'\nu} + (-2p-q)_{\lambda'} g_{\mu\sigma'} + (p+2q)_\mu g_{\lambda'\sigma'}]$$

$$\times\frac{-i\left(g_{\lambda\lambda'}-\dfrac{(1-\beta)q_\lambda q_{\lambda'}}{q^2}\right)}{q^2+i\varepsilon}\frac{-i\left(g_{\sigma\sigma'}-\dfrac{(1-\beta)(p+q)_\sigma(p+q)_{\sigma'}}{(p+q)^2}\right)}{(p+q)^2+i\varepsilon}.$$

The amplitude for Fig. 5.1b is

$$\Pi^{(a)\,ab}_{\mu\nu}(p) = -ig^2\int\frac{d^4q}{(2\pi)^4}\frac{-i(g_{\lambda\lambda'}-(1-\beta)\,q_\lambda q_{\lambda'}/q^2)}{q^2+i\varepsilon}$$

$$\times[\varepsilon^{acf}\varepsilon^{bcf}(g_{\mu\nu}g_{\lambda\lambda'}-g_{\mu\lambda}g_{\nu\lambda'})].$$

There are two identical terms contributing to Fig. 5.1b. But there is also a symmetry factor 1/2.

The amplitudes for the remaining two diagrams are

$$\Pi^{(c)\,ab}_{\mu\nu}(p) = (-1)(-g)^2\int\frac{d^4q}{(2\pi)^4}\varepsilon^{bcd}\varepsilon^{dca}$$

$$\times q_\nu(p+q)_\mu\frac{i}{q^2+i\varepsilon}\frac{-i}{(p+q)^2+i\varepsilon},$$

$$\Pi^{(d)\,ab}_{\mu\nu}(p) = (-1)\int\frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left[(\not{p}+\not{q}+m)\gamma_\mu\frac{\tau^a}{2}(\not{q}+m)\gamma_\nu\frac{\tau^b}{2}\right]$$

$$\times\frac{i}{(p+q)^2-m^2+i\varepsilon}\frac{i}{q^2-m^2+i\varepsilon}.$$

Note that for all amplitudes in these diagrams, we have $\Pi^{ab}_{\mu\nu}\propto\delta^{ab}$ as required by isospin symmetry.

(b) Vertex corrections

There are two vertex diagrams to order $O(g^2)$, as shown in Fig. 5.2.

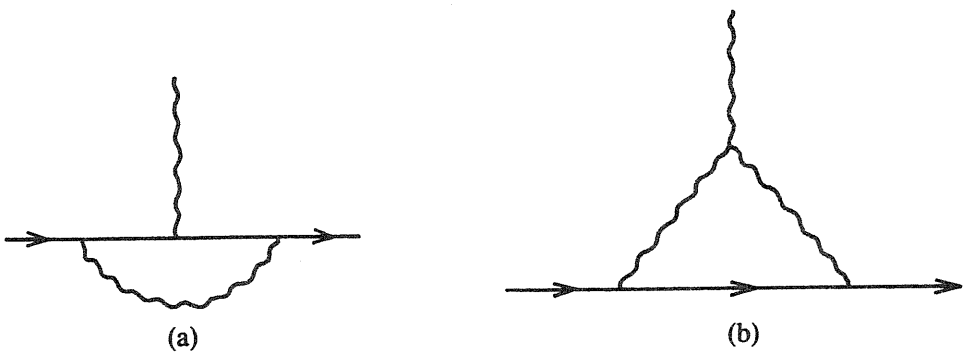


Fig. 5.2 Lowest-order fermion-gauge field vertex diagrams.

It serves as a good practice to write down the amplitudes and carry out the isospin reduction.

Chapter 6

RENORMALIZATION THEORY

We first summarize what we have learned about the renormalization of quantum electrodynamics (QED). We then extend the renormalization to high order processes.

1. Renormalization of QED — One Loop Diagrams

In part I, we have studied a number of one-loop diagrams which are formally divergent. These divergent diagrams and their physical interpretations are

(a) Electron self-energy diagram (Fig. 6.1)

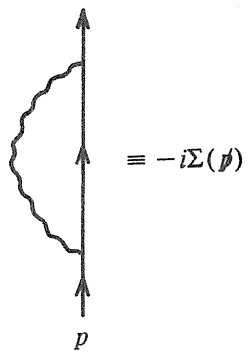


Fig. 6.1 An electron self-energy diagram.

This diagram is linearly divergent. Expanding $\Sigma(\not p)$ around $\not p = m$, we have

$$\Sigma(\not p) = \Sigma(m) + \Sigma'(m)\,(\not p - m) + \tilde{\Sigma}(\not p) \tag{6.1}$$

where

$$\Sigma(m) = \delta m \equiv m - m_0 \approx \frac{3\alpha}{4\pi} m \ln \frac{\Lambda^2}{m^2} \tag{6.2}$$

$$\Sigma'(m) = 1 - Z_2^{-1} \approx -\frac{\alpha}{4\pi}\left(\ln\frac{\Lambda^2}{m^2} - 2\ln\frac{m^2}{\lambda^2}\right) \tag{6.3}$$

are ultraviolet divergent, $\tilde{\Sigma}$ is ultraviolet finite, and $\tilde{\Sigma}(\not{p})=O((\not{p}-m)^2)$ as $\not{p} \to m$.

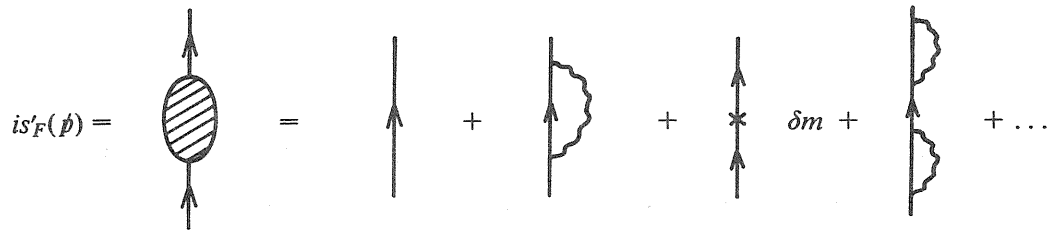


Fig. 6.2 The full electron propagator.

Using Dyson's equation (Fig. 6.2), we have

$$iS'_{\mathrm{F}}(\not{p}) = \frac{i}{\not{p}-m+i\varepsilon} + \frac{i}{\not{p}-m+i\varepsilon}(-i\Sigma(\not{p})+i\delta m)\frac{i}{\not{p}-m+i\varepsilon} + \cdots$$

$$= \frac{i}{\not{p} - m - (\Sigma(\not{p}) - \delta m) + i\varepsilon}$$

$$= Z_2\, i\tilde{S}'_{\mathrm{F}}(\not{p}) + O(\alpha^2) \tag{6.4}$$

where

$$i\tilde{S}'_{\mathrm{F}}(\not{p}) \equiv \frac{i}{\not{p} - m - \tilde{\Sigma}(\not{p}) + i\varepsilon} \tag{6.5}$$

is the finite renormalized electron propagator, accurate to $O(\alpha)$. The divergence of S'_{F} is now isolated into the divergent constant Z_2,

$$Z_2 = 1 - \frac{\alpha}{4\pi}\left(\ln\frac{\Lambda^2}{m^2} - 2\ln\frac{m^2}{\lambda^2}\right) + O(\alpha^2). \tag{6.6}$$

A concise way to describe the physical mass m and the normalization of $\tilde{S}'_{\mathrm{F}}(\not{p})$ is

$$\tilde{S}'^{-1}_{\mathrm{F}}(\not{p})\big|_{\not{p}=m} = 0 \tag{6.7a}$$

$$\frac{d}{d\not{p}}\tilde{S}'^{-1}_{\mathrm{F}}(\not{p})\Big|_{\not{p}=m}=1. \tag{6.7b}$$

(b) Vacuum polarization diagram (Fig. 6.3)

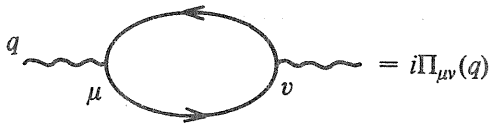


Fig. 6.3 A vacuum polarization diagram.

The vacuum polarization diagram has a superficial quadratic divergence. However, the gauge invariance implies that we can express $\Pi_{\mu\nu}(q)$ as

$$\Pi_{\mu\nu}(q)=(q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi(q^2) \tag{6.8}$$

where $\Pi(q^2)$ only diverges logarithmically. Expanding $\Pi(q^2)$ around $q^2=0$, we have

$$\Pi(q^2)=\Pi(0)+\tilde{\Pi}(q^2) \tag{6.9}$$

with

$$\Pi(0)=Z_3^{-1}-1=\frac{\alpha}{3\pi}\ln\frac{M^2}{m^2} \tag{6.10}$$

being a cutoff-dependent constant. $\tilde{\Pi}(q^2)$ is finite, obeying

$$\tilde{\Pi}(0)=0. \tag{6.11}$$

The full photon propagator (Fig. 6.4) is

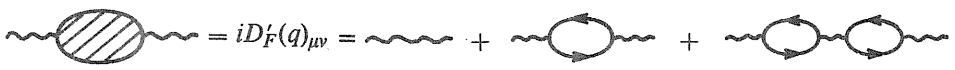


Fig. 6.4 The full photon propagator.

$$\begin{aligned} i\,\tilde{D}'_{\mathrm{F}}(q)_{\mu\nu} &= \frac{-ig_{\mu\nu}}{q^2\,(1+\Pi(q^2))}+q_\mu q_\nu \text{ terms} \\ &= Z_3\,i\tilde{D}'_{\mathrm{F}}(q)_{\mu\nu}+q_\mu q_\nu \text{ terms}+O(\alpha^2) \end{aligned} \tag{6.12}$$

where

$$i\tilde{D}_F(q)_{\mu\nu} = \frac{-ig_{\mu\nu}}{q^2(1+\tilde{\Pi}(q^2))} + q_\mu q_\nu \text{ terms.} \tag{6.13}$$

Here, we have isolated the divergence of $iD'_{F\mu\nu}$ into a single renormalization constant Z_3,

$$Z_3 = 1 - \frac{\alpha}{3\pi} \ln \frac{M^2}{m^2} + O(\alpha^2). \tag{6.14}$$

(c) Vertex correction (Fig. 6.5)

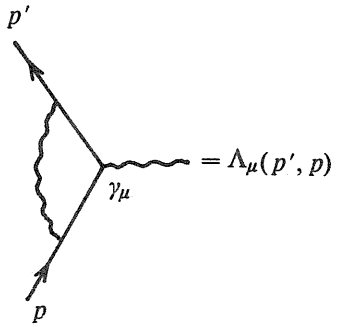


Fig. 6.5 A vertex correction diagram.

The vertex correction diagram diverges logarithmically. The divergent part of Λ_μ is proportional to γ_μ. We can expand Λ_μ as

$$\Lambda_\mu(p', p) = (Z_1^{-1} - 1)\gamma_\mu + \tilde{\Lambda}_\mu(p', p) \tag{6.15}$$

with

$$Z_1^{-1} - 1 = \frac{\alpha}{4\pi}\left(\ln \frac{\Lambda^2}{m^2} - 2 \ln \frac{m^2}{\lambda^2}\right). \tag{6.16}$$

All cutoff dependence is in Z_1. $\tilde{\Lambda}_\mu \equiv$ finite and may be determined uniquely by

$$\bar{u}(p)\,\tilde{\Lambda}_\mu(p, p)\,u(p) = 0 \tag{6.17}$$

$$(\not{p} - m)\,u(p) = 0. \tag{6.18}$$

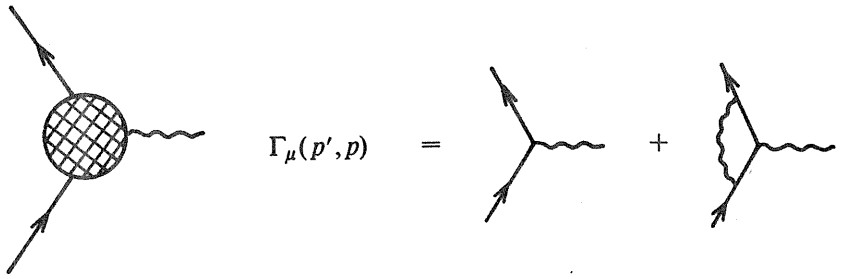


Fig. 6.6 The full vertex diagram.

Then, the full vertex Γ_μ in Fig. 6.6 is

$$\begin{aligned}\Gamma_\mu(p',p) &= \gamma_\mu + \Lambda_\mu(p',p) \\ &= Z_1^{-1}\tilde{\Gamma}_\mu(p',p) + O(\alpha^2)\end{aligned} \tag{6.19}$$

where the finite renormalized vertex is

$$\tilde{\Gamma}_\mu(p',p) = \gamma_\mu + \tilde{\Lambda}_\mu(p',p). \tag{6.20}$$

The divergence of Γ_μ is absorbed into the renormalization constant Z_1,

$$Z_1 = 1 - \frac{\alpha}{4\pi}\left(\ln\frac{\Lambda^2}{m^2} - 2\ln\frac{m^2}{\lambda^2}\right) + O(\alpha^2). \tag{6.21}$$

Comparing (6.6) and (6.21), we recognize that

$$Z_1 = Z_2. \tag{6.22}$$

As we all know, Eq. (6.22) is a general result. It follows from the Ward identity

$$(p'-p)_\mu \Gamma^\mu(p'-p) = S_F'^{-1}(\not{p}') - S_F'^{-1}(\not{p}) \tag{6.23}$$

$$(p'-p)_\mu \tilde{\Gamma}^\mu(p'-p) = \tilde{S}_F'^{-1}(\not{p}') - \tilde{S}_F'^{-1}(\not{p}). \tag{6.24}$$

We can express the Ward identity either in terms of the unrenormalized quantities (Eq. (6.23)) or in terms of the renormalized quanties (Eq. (6.24)).

We have also learned in Part I that only the combination

$$e = e_0 Z_1^{-1} Z_2 \sqrt{Z_3} = e_0 \sqrt{Z_3} \tag{6.25}$$

enters into the scattering amplitudes. We identify e as the physical charge of an electron. The bare charge, e_0, is a cutoff-dependent quantity, and its magnitude depends on the ultraviolet resolution.

2. Dyson's Prescriptions

We first follow the historical path and describe the Dyson's prescriptions. Dyson introduced the concepts of skeletons and insertions as follows.

(i) Insertions: All self-energy parts and vertex corrections are called insertions.

(ii) Skeletons: Diagrams with all insertions removed are called skeletons.

We use several examples to illustrate the definitions.

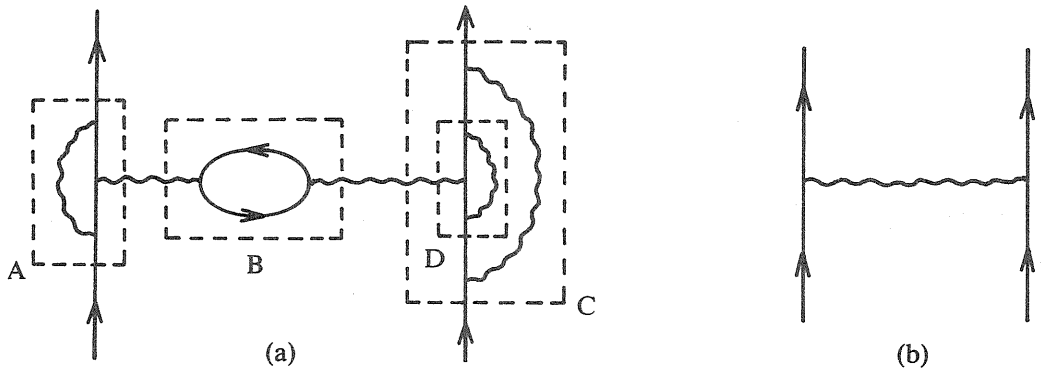


Fig. 6.7 (a) a diagram with insertions, (b) the skeleton diagram.

In Fig. 6.7a, parts A, B, C, and D are insertions. In particular, part D is an insertion to an insertion. Fig. 6.7b is the corresponding skeleton diagram.

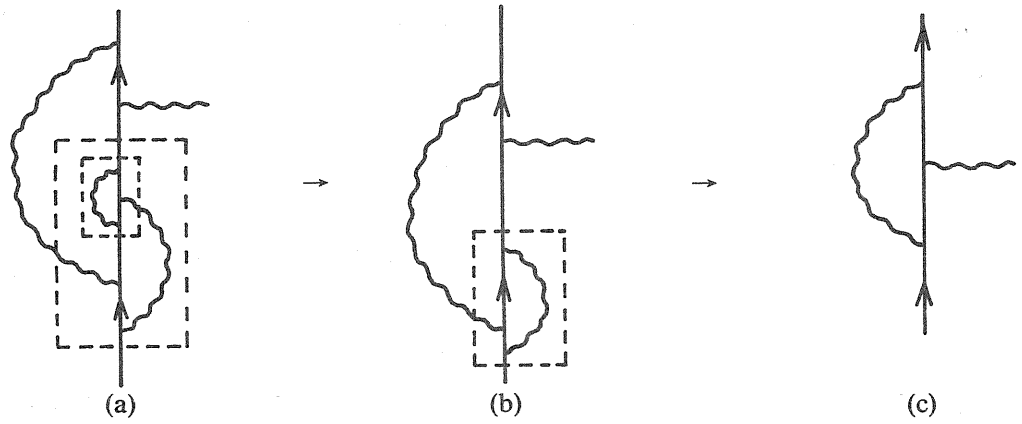


Fig. 6.8 The reduction of a nested insertion diagram to its skeleton diagram.

Fig. 6.8a is an example of nested insertions. After we remove the successive insertions, we arrive at the skeleton diagram Fig. 6.8c.

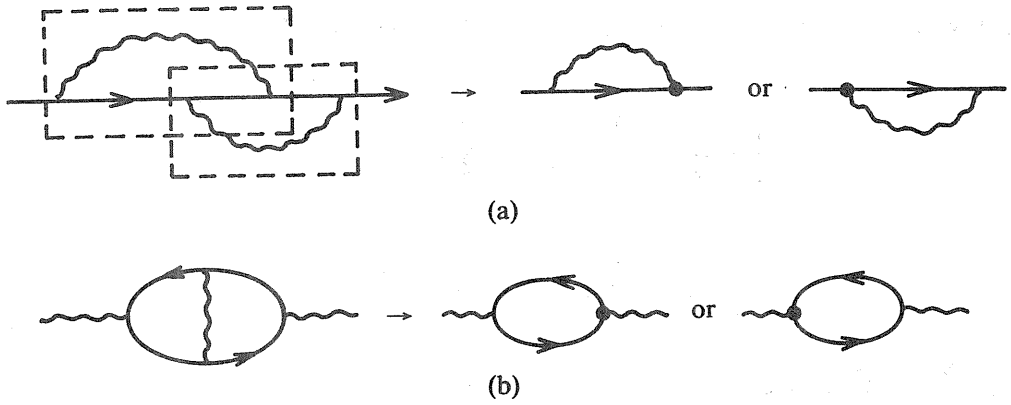


Fig. 6.9 Diagrams with overlapping insertions.

Fig. 6.9 gives examples of overlapping insertions (divergences). In Fig. 6.9a and b, there are two different ways of removing the insertions. We shall postpone the discussion of overlapping divergences to the next section. Dyson's prescriptions work well when there are no overlapping divergences.

Dyson's prescriptions are:

(a) Group diagrams with the same skeleton but different insertions together;
(b) Computing the sum of these diagrams by using the following Feynman rules on skeleton diagrams only: $-ie\tilde{\Gamma}_\mu$ as the vertices, $i\tilde{S}'_F$ and $i\tilde{D}'_F$ as the propagators;
(c) Inserting the free particle wave functions,

$$\frac{\varepsilon_\mu(k,\lambda)}{\sqrt{2k(2\pi)^3}}, \quad \sqrt{\frac{m}{E(2\pi)^3}}\, u(p,s), \text{ etc.}$$

for the appropriate external particles.

We use Dyson's prescriptions to compute the renormalized propagators and vertices iteratively. We have already known how to compute $\tilde{S}'_F$, $\tilde{D}'_F$, and $\tilde{\Gamma}_\mu$ to $O(\alpha)$. Using these propagators and vertices, we can compute $\tilde{\Sigma}$, $\tilde{\Pi}$, and $\tilde{\Lambda}_\mu$ to $O(\alpha^2)$, etc.

To demonstrate the method, we consider the electron-electron scattering to $O(\alpha^2)$ in Fig. 6.10.

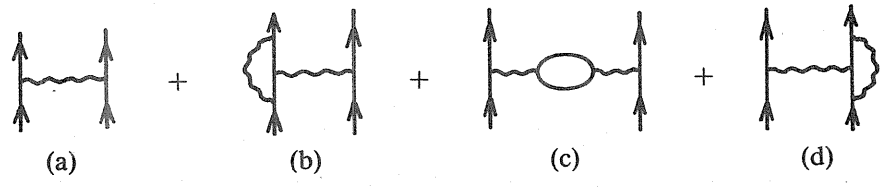
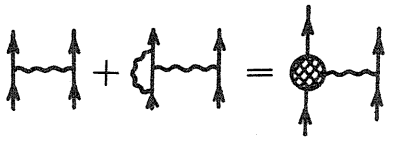


Fig. 6.10 Electron-electron scattering amplitude plus $O(\alpha)$ insertions.

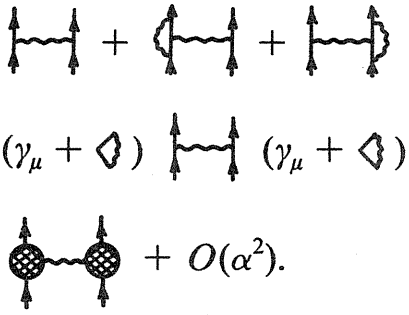
We can write the sum of amplitudes (a) and (b) as



$$= (\gamma_\mu + \Lambda_\mu)\ \gamma_\nu \qquad (6.26)$$

where Λ_μ is the shorthand notation of Λ_μ.

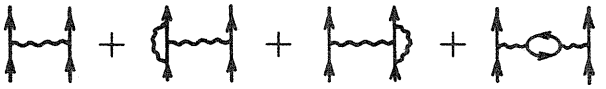
To order $O(\alpha^2)$, we can write the sum of diagrams (a), (b), and (c) as

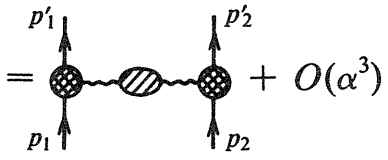


$$= (\gamma_\mu + \Lambda_\mu)\ (\gamma_\mu + \Lambda_\mu) + O(\alpha^2)$$

$$= \quad + O(\alpha^2). \qquad (6.27)$$

It is now clear that we can sum all diagrams in Fig. 6.10 to give





$$+ O(\alpha^3)$$

$$= \bar{u}(p_1')\,(-ie_0)\,\Gamma_\mu(p_1', p_1)\,u(p_1)\,\bar{u}(p_2')\,(-ie_0)\,\Gamma_\nu(p_2', p_2)\,u(p_2)$$

$$\times\, iD_F'(p_1' - p_1)_{\mu\nu}$$

$$= Z_2^{-2}\Big\{\bar{u}(p_1')\left(-ie_0\, Z_1^{-1}\, Z_2\, \sqrt{Z_3}\right)\tilde{\Gamma}_\mu(p_1', p_1)\,u(p_1)$$

$$\times\, \bar{u}(p_2')\left(-ie_0\, Z_1^{-1}\, Z_2\, \sqrt{Z_3}\right)\tilde{\Gamma}_\nu(p_2', p_2)\,u(p_2)$$

$$\times\, i\tilde{D}_F(p_1' - p_1)_{\mu\nu}\Big\}. \qquad (6.28)$$

The expression in the { } is precisely the finite part given by the Dyson's prescriptions. To obtain the finite expression, we need to associate a $\sqrt{Z_2}$ factor for each of the external electrons, and a $\sqrt{Z_3}$ factor for

each of the external photons. To understand these $\sqrt{Z}$ factors, we can visualize the physical electron as an electron emitted and absorbed with a large space-time separation (see Fig. 6.11).

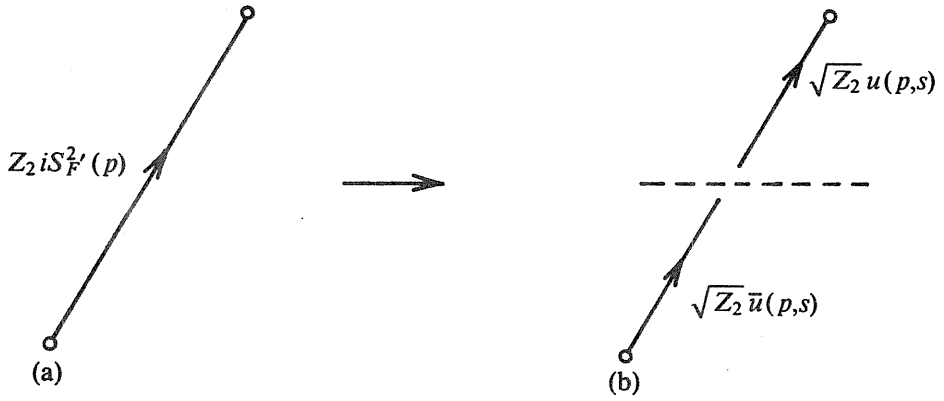


Fig. 6.11 The emission and absorption of a physical electron may be interpreted as the propagation of a virtual particle at a large space-time separation.

The factor Z_2 associated with the propagator in Fig. 6.11a becomes two $\sqrt{Z_2}$ factors associated with the external particles in Fig. 6.11b.

To compute the fourth order ($O(\alpha^2)$) vertices, we need to consider two skeleton diagrams in Fig. 6.12.

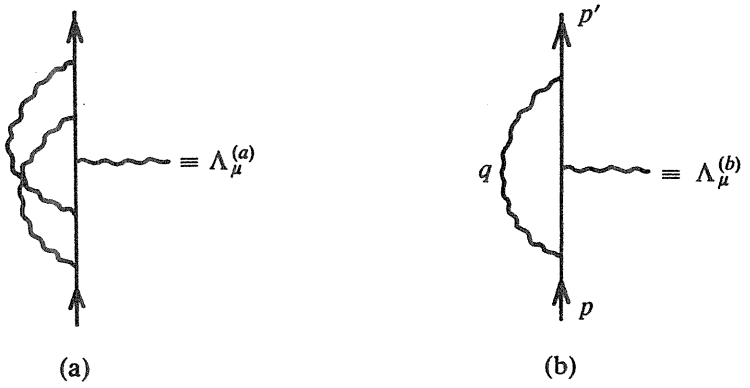


Fig. 6.12 Two skeleton diagrams appearing in the fourth order vertex calculation.

Figure 6.12a is a fourth order renormalization part by itself. We do not need to consider any insertion to it. To compute $\Lambda_\mu^{(b)}$ of Fig. 6.12b to fourth order in e, we need to include all second order insertions as shown in Fig. 6.13, giving

$$\Lambda_\mu^{(b)}(p', p) = \int \frac{d^4q}{(2\pi)^4}(-ie)\,\tilde{\Gamma}_\lambda(p', p'-q)$$

$$\times\, i\tilde{S}'_{\mathrm{F}}(\not{p}' - \not{q})\,\tilde{\Gamma}_{\mu}(p'-q, p-q)\, i\tilde{S}'_{\mathrm{F}}(\not{p}'\not{q})$$

$$\times\, (-ie)\,\tilde{\Gamma}_{2}(p-q, p)\, i\tilde{D}'_{\mathrm{F}}(q)_{\lambda\nu}. \tag{6.29}$$

We can show that both $\Lambda_{\mu}^{(a)}$ and $\Lambda_{\mu}^{(b)}$ are logarithmically divergent, and

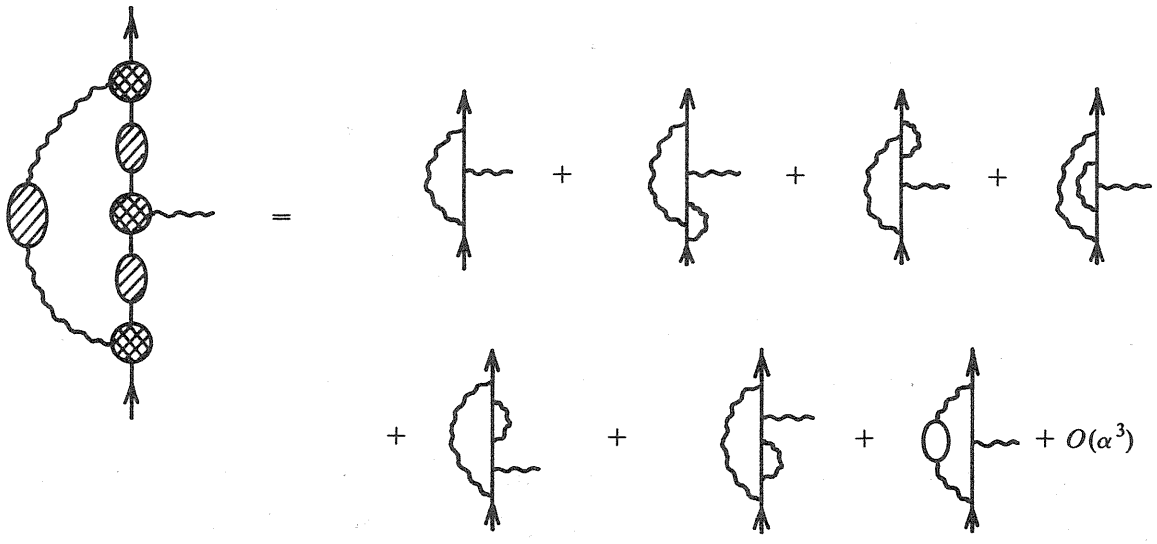


Fig. 6.13 Second order insertions to diagram in Fig. 6.12b.

that the divergent parts are proportional to γ_{μ}. Thus, we can introduce

$$\Lambda_{\mu}^{(a)} = \delta(Z_1^{-1})^{a}\,\gamma_{\mu} + \tilde{\Lambda}_{\mu}^{(a)} \tag{6.30a}$$

$$\Lambda_{\mu}^{(b)} = \delta(Z_1^{-1})^{b}\,\gamma_{\mu} + \tilde{\Lambda}_{\mu}^{(b)}\,. \tag{6.30b}$$

Adding $\tilde{\Lambda}_{\mu}^{(a)}$ and $\tilde{\Lambda}_{\mu}^{(b)}$ to the second order vertex $\tilde{\Gamma}_{\mu}$, we obtain the renormalized vertex $\tilde{\Gamma}_{\mu}(p', p)$ with all $O(\alpha^2)$ contributions.

Dyson's prescriptions work as long as there are no overlapping divergences.

3. Overlapping Divergences

In QED, only electron and photon self-energy diagrams may have overlapping divergences. There are no overlapping divergences in vertex corrections. For this reason, we can construct $\tilde{\Gamma}_{\mu}(p', p)$ to any given order n if we know $\tilde{S}'_{\mathrm{F}}$, $\tilde{D}'_{\mathrm{F}}$, and $\tilde{\Gamma}_{\mu}$ to order $n-1$ as follows. Using the Ward identity,

$$(p'-p)_\mu \tilde{\Gamma}^\mu(p', p) = \tilde{S}_F'^{-1}(\not{p}') - \tilde{S}_F'^{-1}(\not{p}),$$

we can obtain

$$\tilde{S}_F'^{-1}(\not{p}') = (p'-p)_\mu \tilde{\Gamma}_\mu(p', p)\big|_{\not{p}=m} \tag{6.31}$$

to order n as desired. Thus, the real difficulty is in the construction of the photon self-energy.

Consider the overlapping diagram in Fig. 6.14.

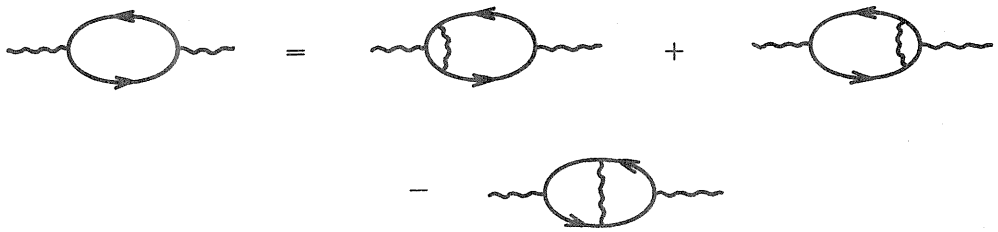


Fig. 6.14 Overcounting and compensation.

Bjorken and Drell said that we should make vertex corrections at both vertices. This leads to overcounting. They suggested that one can correct this overcounting by subtracting out an electron-positron scattering kernel, as in Fig. 6.15. One then inserts $\tilde{\Gamma}_\mu$, $\tilde{S}_F'$, and $\tilde{D}_F'$ into the "blobs" in Fig. 6.15.

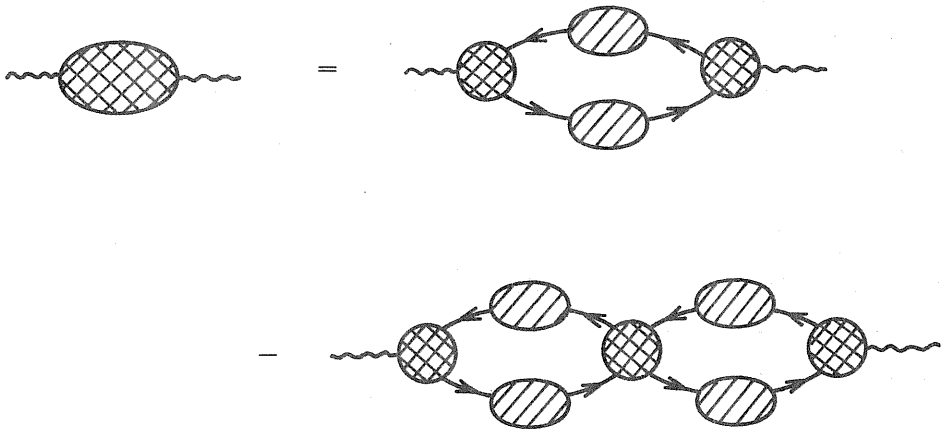


Fig. 6.15 Generalization of Fig. 6.14 to all orders.

It provides a unique way of factoring out the divergences. One can verify the expression in Fig. 6.15 by working out a few diagrams explicitly.

The above method of removing overlapping divergences is not satisfactory because it cannot be generalized to other theories in a straightforward way. However, this method does work in QED. In the next section, we shall describe a more general method of handling overlapping divergences.

4. BPHZ Renormalization

We describe here a renormalization method developed and perfected by Bogoliubov, Parasiuk, Hepp, and Zimmermann (BPHZ renormalization). In order to understand the BPHZ renormalization, we need to introduce their terminology first.

(a) Subgraph
A subgraph is defined by a set of vertices and all the lines connecting these vertices.

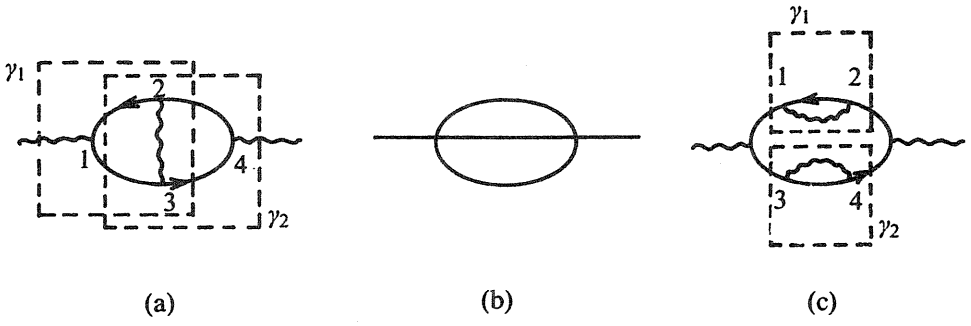


Fig. 6.16 Examples of subgraphs.

For instance, Fig. 6.16a contains two non-trivial subgraphs defined by vertices $\gamma_1 \equiv (1,2,3)$ and $\gamma_2 \equiv (2,3,4)$. Fig. 6.16(b) contains no non-trivial subgraphs. Fig. 6.16c contains two non-trivial subgraphs $\gamma_1 \equiv (1,2)$ and $\gamma_2 \equiv (3,4)$.

(b) Disjoint subgraphs
Two diagrams γ_1 and γ_2 are disjoint if there are no vertices in common. We denote this relation by $\gamma_1 \bigcap \gamma_2 = \varnothing$.

For instance, the two subgraphs γ_1 and γ_2 in Fig. 6.16c are disjoint.

(c) Non-Overlapping graphs
Two graphs γ_1 and γ_2 are non-overlapping if any of the following three conditions is satisfied. (1) $\gamma_1 \subset \gamma_2$, (2) $\gamma_2 \subset \gamma_1$, and (3) $\gamma_1 \bigcap \gamma_2 = \varnothing$. Otherwise, γ_1 and γ_2 are overlpping.

For instance, γ_1 and γ_2 defined in Fig. 6.17a and Fig. 6.17b are non-

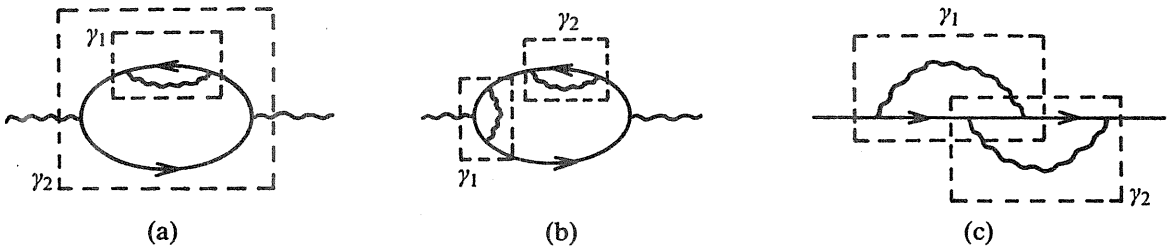


Fig. 6.17 Possible relations between two subgraphs.

overlapping, but γ_1 and γ_2 defined in Fig. 6.17c and Fig. 6.16a are overlapping.

(d) Unrenormalized and renormalized amplitudes

We consider a diagram Γ and their internal integrations $\prod_i \frac{d^4k_i}{(2\pi)^4}$. The unrenormalized amplitude is

$$M_\Gamma = \int \prod_i \frac{d^4k_i}{(2\pi)^4} I_\Gamma(\{p\}, \{k\}) \tag{6.32a}$$

where $\{p\}$ denotes the external momenta, and $\{k\}$ denotes the internal momentum variables. We obtain M_Γ from Γ by the usual Feynman rules without any subtractions. We call I_Γ the unrenormalized integrand. After the renormalization, we write

$$\tilde{M}_\Gamma = \int \prod_i \frac{d^4k_i}{(2\pi)^4} \tilde{I}_\Gamma(\{p\}, \{k\}). \tag{6.32b}$$

We call $\tilde{M}_\Gamma$ and $\tilde{I}_\Gamma$ the renormalized amplitude and the renormalized integrand, respectively. Our job is to construct $\tilde{I}_\Gamma$ from I_Γ.

BPHZ prescriptions are

(a) Diagrams without divergent subgraphs

We denote the superficial degrees of divergence of Γ by $D(\Gamma)$.

If Γ is not a renormalization part (i.e. $D(\Gamma) < 0$). we define

$$\tilde{I}_\Gamma = I_\Gamma. \tag{6.33}$$

If Γ is a renormalization part ($D(\Gamma) \geqslant 0$), we define

$$\tilde{I}_\Gamma = R_\Gamma I_\Gamma \equiv (1 - t^\Gamma) I_\Gamma \tag{6.34}$$

where

$$t^{\Gamma} f(p_1, \cdots, p_n) \equiv f(p_1^0, \cdots, p_n^0)$$

$$+ \cdots + \frac{1}{d!} \Sigma (p-p^0)_i \cdots (p-p^0)_j \frac{\partial^d f(\{p\})}{\partial p_i \cdots \partial p_j}$$

$$= \text{first } d\text{-derivative terms in the Taylor's expansions of } f(\{p\}) \tag{6.35}$$

with

$$d \equiv \Gamma(d).$$

The point $\{p^0\}$ about which we make the Taylor's expansions is called the renormalization point. In (6.34) and (6.35), t^{Γ} is a shorthand of $t^{D(\Gamma)}$. Thus, $(1-t^{\Gamma})$ describes that the first $D(\Gamma)$ derivative terms are removed in the Taylor's expansion. It is easy to see that $\tilde{M}_{\Gamma}$ is finite by construction. We can check that this construction reproduces the finite parts $\tilde{\Sigma}$, $\tilde{\Pi}$, and $\tilde{\Lambda}_{\mu}$ of the second order QED calculations.

(b) Diagrams with no overlapping divergences

For this class of diagrams, we have

$$\tilde{I}_{\Gamma} = \prod_i (1 - t^{\gamma_i}) I_{\Gamma} \tag{6.36}$$

where $\{\gamma_i\}$ $(i=1, \cdots, n)$ describes all renormalization parts including the graph itself. This prescription is equivalent to the Dyson's prescriptions studied earlier.

We can expand (6.36) to give

$$\tilde{I}_{\Gamma} = \sum_U \prod_{\gamma \in U} (- t^{\gamma}) I_{\Gamma} \tag{6.37}$$

where

$$U = \varnothing; \gamma_1; \cdots; \gamma_u; \{\gamma_1, \gamma_2\}; \cdots; \{\gamma_1, \cdots, \gamma_u\} \tag{6.38}$$

is the collection of all possible divergent subgraphs. We call $\{\gamma_1, \cdots, \gamma_m\}$ a forest made of non-overlapping elements $\gamma_1, \cdots, \gamma_m$. We shall give the definition of a general forest in the next section.

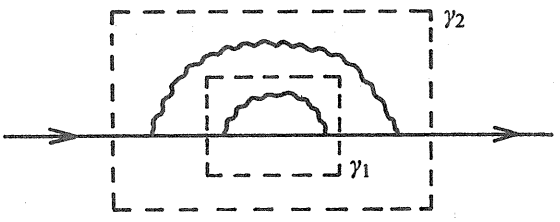


Fig. 6.18 A self-energy diagram with 2 nonoverlapping renormalization parts.

For instance, in Fig. 6.18, there are two renormalization parts denoted by subgraphs γ_1 and γ_2. The renormalized amplitude is

$$\tilde{I}_\Gamma = (1 - t^{\gamma_1})(1 - t^{\gamma_2}) I_\Gamma$$

$$= I_\Gamma + (-t^{\gamma_1}) I_\Gamma + (-t^{\gamma_2}) I_\Gamma + (-t^{\gamma_1})(-t^{\gamma_2}) I_\Gamma.$$

The forests U include

$$U = \varnothing;\ \gamma_1;\ \gamma_2;\ \{\gamma_1, \gamma_2\}.$$

(c) Overlapping diagrams

Equation (6.37) still holds in the presence of overlapping divergences. Not all renormalization parts may be put together to make a forest. A forest U belonging to a diagram Γ is a hierarchy of subgraphs such that

(i) Elements of U are renormalization parts;
(ii) Any two elements $\gamma_1, \gamma_2 \in U$ are non-overlapping;
(iii) $U=\varnothing$ is a forest.

Diagram Γ in Fig. 6.19 contains overlapping subdiagrams γ_1 and γ_2.

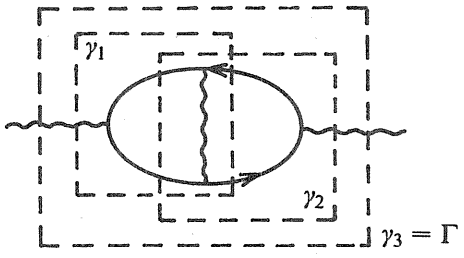


Fig. 6.19 A vacuum polarization diagram with overlapping subdiagrams.

All possible forests include

$$U = \varnothing;\ \gamma_1;\ \gamma_2;\ \gamma_3;\ \{\gamma_1, \gamma_3\};\ \{\gamma_2, \gamma_3\}. \tag{6.39}$$

Note that $\{\gamma_1, \gamma_2\}$ and $\{\gamma_1, \gamma_2, \gamma_3\}$ are not forests. According to Eq. (6.37), the renormalized integrand $\tilde{I}_\Gamma$ is

$$\tilde{I}_\Gamma = \sum_{\text{all } U} \prod_{\gamma \in U} (-t^\gamma)\, I_\Gamma$$

$$= I_\Gamma + (-t^{\gamma_1})\, I_\Gamma + (-t^{\gamma_2})\, I_\Gamma + (-t^{\gamma_3})\, I_\Gamma$$

$$+ (-t^{\gamma_3})(-t^{\gamma_1})\, I_\Gamma + (-t^{\gamma_3})(-t^{\gamma_2})\, I_\Gamma$$

$$= (1 - t^{\gamma_3})(1 - t^{\gamma_1} - t^{\gamma_2})\, I_\Gamma \tag{6.40a}$$

$$\neq (1 - t^{\gamma_3})(1 - t^{\gamma_1})(1 - t^{\gamma_2})\, I_\Gamma . \tag{6.40b}$$

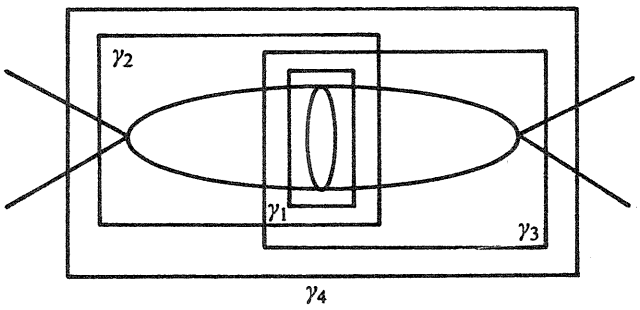


Fig. 6.20 A 4-point vertex diagram in ϕ^4, theory with overlapping subdiagrams.

As a final example, we consider the overlapping divergent diagram Fig. 6.20 in the ϕ^4 theory. Note that

$$\gamma_4 \supset \gamma_2, \gamma_3 \supset \gamma_1 \quad \text{and} \quad \gamma_2 \cap \gamma_3 \neq \varnothing.$$

The possible forests are

$$U = \varnothing; \gamma_1; \gamma_2; \gamma_3; \gamma_4; \{\gamma_1, \gamma_2\}; \{\gamma_1, \gamma_3\};$$

$$\{\gamma_1, \gamma_4\}; \{\gamma_2, \gamma_4\}; \{\gamma_3, \gamma_4\}; \{\gamma_1, \gamma_2, \gamma_4\};$$

$$\{\gamma_1, \gamma_3, \gamma_4\}. \tag{6.41}$$

The renormalized integrand is

$$\tilde{I}_\Gamma = \big(1 - t^{\gamma_1} - t^{\gamma_2} - t^{\gamma_3} - t^{\gamma_4} + \big(-t^{\gamma_1}\big)\big(-t^{\gamma_2}\big)$$
$$+ \big(-t^{\gamma_1}\big)\big(-t^{\gamma_3}\big) + \big(-t^{\gamma_1}\big)\big(-t^{\gamma_4}\big) + \big(-t^{\gamma_2}\big)\big(-t^{\gamma_4}\big)$$
$$+ \big(-t^{\gamma_3}\big)\big(-t^{\gamma_4}\big) + \big(-t^{\gamma_1}\big)\big(-t^{\gamma_2}\big)\big(-t^{\gamma_4}\big) + \big(-t^{\gamma_1}\big)\big(-t^{\gamma_3}\big)\big(-t^{\gamma_4}\big)\big) I_\Gamma$$
$$= \big(1 - t^{\gamma_4}\big)\big(1 - t^{\gamma_2} - t^{\gamma_3}\big)\big(1 - t^{\gamma_1}\big) I_\Gamma. \tag{6.42}$$

Since γ_1 and γ_4 do not overlap with any subgraphs, they factor out according to Dyson's prescriptions. The only modification in $\tilde{I}_\Gamma$ is due to γ_2 and γ_3.

5. Classification of Renormalizable Theories

If we write down a Lagrange function, how do we know whether the theory is renormalizable? A simple criterion is the following: For all classes of diagrams with fixed numbers of external legs, if the degrees of divergences D do not increase as the number of internal vertices increases, the theory is renormalizable. If the D's for some classes of diagrams increase as V increases, the theory is non-renormalizable. Within the renormalizable theories, if all D's actually decrease as V increases, the theory is called super-renormalizable.

Consider a general interaction

$$\mathscr{L}_{\text{int}} = \sum_a \mathscr{L}_a(\psi, \bar{\psi}, \phi, A_\mu \text{ or } B_\mu) \tag{6.43}$$

consisting of scalar fields ϕ, Dirac fields ψ, and gauge fields A_μ or B_μ.Let

b_a be the number of bosons in $\mathscr{L}_a$,
f_a be the number of fermions in $\mathscr{L}_a$, and
k_a be the number of derivatives in $\mathscr{L}_a$.

Now consider a typical graph. Let

$F_i(F_e)$ be the total number of internal (external) fermion lines,
$B_i(B_e)$ be the total number of internal (external) boson lines,
V be the total number of the vertices,
k be the total number of the derivatives, and
V_a be the number of the a-th vertices (i.e. $\mathscr{L}_a$).

We associate the internal propagators and vertices with the following effective momentum powers:

$$\text{Fermion propagator} \to \frac{d^4p}{(2\pi)^4}\frac{i}{\not{p}-m+i\varepsilon} = O(p^3)$$

$$\text{Boson propagator} \to \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon} = O(p^2)$$

$$\text{Vertex} \to (2\pi)^4\,\delta^4(\Sigma p) = O(p^{-4})$$

$$\text{Derivative } \partial \to p = O(p).$$

Thus the superficial degrees of divergence of the above diagram is

$$D = 3F_i + 2B_i - 4V + 4 + k \tag{6.44}$$

with

$$k = \Sigma V_a k_a \tag{6.45a}$$

$$V = \Sigma V_a \tag{6.45b}$$

$$2F_i + F_e = \Sigma V_a f_a \tag{6.45c}$$

$$2B_i + B_e = \Sigma V_a b_a. \tag{6.45d}$$

Eliminating F_i, B_i, V, and k, we have

$$D = \sum_a V_a\left(\frac{3}{2}f_a + b_a - 4 + k_a\right) - \frac{3}{2}F_e - B_e + 4. \tag{6.46}$$

In order that D does not increase with V, we require for renormalizable theories

$$d_a \equiv \frac{3}{2}f_a + b_a - 4 + k_a \leqslant 0. \tag{6.47}$$

In Table VI, we summarize all the possible renormalizable (including super-renormalizable) theories in 3+1 dimensions.

Table VI Summary of renormalizable theories in 3 + 1 dimensions.

f_a	b_a	k_a	d_a	Remarks or examples
0	1	0	-3	$\lambda\phi$, tadpole
0	1	1, 2, 3	$-2, -1, 0$	$\lambda\partial_\mu B_\mu$ etc. integrated out
0	2	0	-2	$\mu^2\phi^2$, boson mass term
0	2	1	-1	$A_\mu\partial_\mu\phi$, Higgs mechanism
0	2	2	0	$(\partial_\mu\phi)^2$, $(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$, kinetic term
0	3	0	-1	$\lambda\phi^3$
0	3	1	0	$A_\mu\phi^*\partial_\mu\phi$, $\mathbf{B}_\mu \times \mathbf{B}_\nu \cdot (\partial_\mu \mathbf{B}_\nu - \partial_\nu \mathbf{B}_\mu)$
0	4	0	0	$\lambda\phi^4$, $(\mathbf{B}_\mu \times \mathbf{B}_\nu)^2$
2	0	0	-1	$m\bar{\psi}\psi$, fermion mass term
2	0	1	0	$\bar{\psi}\partial\!\!\!/\psi$ fermion kenetic term
2	1	0	0	$\bar{\psi}\gamma_5\psi\phi$, $\bar{\psi}\gamma_\mu\psi A_\mu$

For a massive vector field ϕ_μ, the propagator goes as

$$\frac{-i(g_{\mu\nu} - p_\mu p_\nu/m^2)}{p^2 - m^2 + i\varepsilon} \to O(1).$$

It is easy to see that, by power counting alone, $\bar{\psi}\gamma_\mu\psi\phi_\mu$ is no longer renormalizable. However, if we begin with a renormalizable gauge theory and let the mass of a guage field be generated spontaneously, then, the theory remains to be renormalizable. This is known as the Higg's mechanism and plays an important role in the Weinberg-Salem theory of electro-weak interaction.

Chapter 7

RENORMALIZATION OF NON-ABELIAN GAUGE THEORIES

1. Non-Abelian Gauge Theories

The generalization of a Yang-Mills theory to an arbitrary non-abelian gauge theory is straightforward.

We consider a Lie group G generated by the algebra

$$[X^a, X^b] = if^{abc}X^c \tag{7.1}$$

where X^a is the infinitesimal generator of the Lie group. We refer f^{abc} as the structure constants of the group. For simplicity, we restrict ourselves to group SU(N). With proper choices of X^a, we can make f^{abc} to be totally antisymmetric. We call a set of $N \times N$ matrices T^a a representation of the algebra if T^a obeys the same set of algebraic equations as (7.1),

$$[T^a, T^b] = if^{abc}T^c. \tag{7.2}$$

For group SU(N), we may choose T to be hermitian.

We introduce a set of gauge fields B_μ^a associated with the generators of G. We also introduce a set of fermion fields $\psi(x)$ which transform locally under G as

$$\psi(x) \to \psi'(x) = U(g)\,\psi(x) \tag{7.3}$$

where

$$U(g) = \exp(-ig\omega^a T^a) \tag{7.4}$$

is an $N \times N$ matrix whose infinitesimal generator is T. The gauge field tensor is

$$G_{\mu\nu}^a \equiv \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g f^{abc} B_\mu^b B_\nu^c. \tag{7.5}$$

The Lagrange function describing a non-abelian gauge field interacting with a Dirac field is

$$\mathscr{L} = -\frac{1}{4}(G^a_{\mu\nu})^2 + \bar{\psi}\left[i\gamma_\mu(\partial_\mu - igT^a B^a_\mu) - m\right]\psi. \tag{7.6}$$

The infinitesimal gauge transformations on ψ, B^a_μ, $G^a_{\mu\nu}$, and $\mathscr{L}$ are

$$\delta\psi = \psi'(x) - \psi(x) = -ig\delta\omega^a T^a \psi(x) \tag{7.7}$$

$$\begin{aligned}\delta B^a_\mu &\equiv B^{a'}_\mu(x) - B^a_\mu(x) \\ &= -\partial_\mu \delta\omega^a + g f^{abc}\delta\omega^b B^c_\mu\end{aligned} \tag{7.8}$$

$$\delta G^a_{\mu\nu} = g f^{abc}\delta\omega^b G^c_{\mu\nu} \tag{7.9}$$

and

$$\delta\mathscr{L} = 0 \tag{7.10}$$

where $\delta\omega^a$ is the infinitesimal gauge parameter. The Lagrange function is invariant under the local gauge transformation.

To gain some feeling about f^{abc} and T^a. We review the group SU(3). The physicists usually introduce the SU(3) algebra as those generated by the following 3×3 traceless, hermitian matrices, known as the Gell-Mann matrices,

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The matrices λ's obey the normalization condition

$$\mathrm{Tr}\,(\lambda^a\lambda^b) = 2\delta^{ab}. \tag{7.11}$$

We define the SU(3) algebra and the structure constants by

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = if^{abc}\frac{\lambda^c}{2}. \tag{7.12}$$

From (7.11) and (7.12), we have

$$\begin{aligned} if^{abc} &= \mathrm{Tr}\left(\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right]\lambda^c\right) \\ &= \frac{1}{4}\mathrm{Tr}\,(\lambda^a\lambda^b\lambda^c - \lambda^b\lambda^a\lambda^c). \end{aligned} \tag{7.13}$$

Using the cyclic property of traces, we can verify easily that f^{abc} is indeed totally antisymmetric. Another universal property of the Lie algebra is that the structure constants themselves form a representation (adjoint representation) of the group. We may introduce the adjoint representation by the matrices

$$(M^b)_{ac} \equiv if^{abc}. \tag{7.14}$$

By applying the Lie algebra to the Jacobi identity

$$[[X^a, X^b], X^c] + [[X^b, X^c], X^a], [[X^c, X^a], X^b] = 0, \tag{7.15}$$

we can verify that

$$[M^a, M^b] = if^{abc}M^c. \tag{7.16}$$

The dimension of the adjoint representation is equal to the number of the generators of the Lie group. For group SU(N), it describes a $(N^2-1) \times (N^2-1)$ representation. We can rewrite (7.9) as

$$\begin{aligned} \delta G^a_{\mu\nu} &= -igif^{abc}\delta\omega^b G^c_{\mu\nu} \\ &= -ig\delta\omega^b\,(M^b G_{\mu\nu})^a \end{aligned}$$

which has the same form as (7.7).

If we choose the group G to be SU(3), and ψ to be sets of three dimensional (SU(3) spinor) representations of SU(3), we arrive at the Lagrange function for the quantum chromodynamics,

$$\mathscr{L} = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} + \sum_f \bar{\psi}_f\left[i\gamma_\mu\left(\partial_\mu - ig\frac{\lambda^a}{2}B^a_\mu\right) - m_f\right]\psi_f \quad (7.17)$$

where the index f describes the flavor of the quark field. At the moment, we believe that there are at least six flavors of quarks, known as up, down, strange, charm, top, and bottom (u, d, s, c, t, b). The gauge fields are known as gluons, and the symmetry is known as the color symmetry. In the standard grand unification theory, the gauge group is extended to SU(5).

For an arbitrary non-abelian gauge group G with the structure constant f^{abc} and the fermion representation T^a, we can obtain the Feynman rules from the Yang-Mills theory by changing ε^{abc} to f^{abc}, and $\tau^a/2$ to T^a. The summation over a runs over the number of generators of G.

2. Gauge Field Self-Energy Diagrams

We consider the one-loop contribution to the Yang-Mills gauge field self-energy diagrams as shown in Fig. 7.1. (The explicit expressions are given in Chap. 5, Sec. 5.)

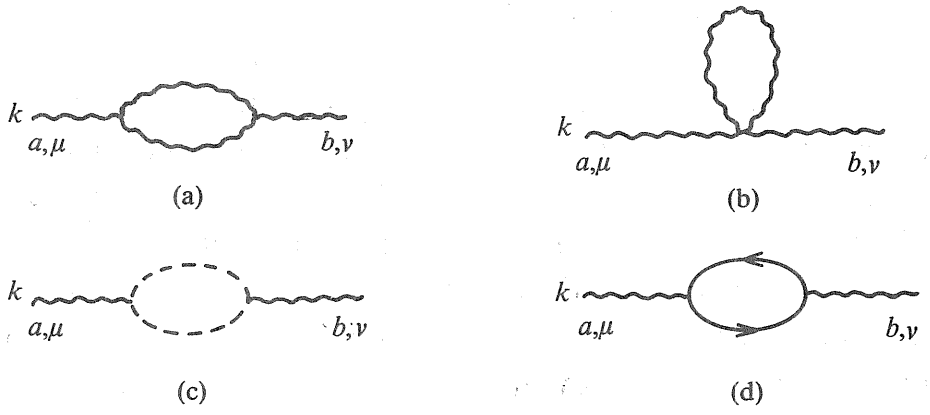


Fig. 7.1 (a)–(d) One-loop contributions to the Yang-Mills gauge-field self-energy diagrams.

The contribution due to Fig. 7.1d is the QED result with $e \to g$ and then multiplied by $\frac{1}{2}\delta^{ab}$. We know how to handle this diagram.

The graphs 7.1a–c are purely Yang-Mills diagrams. We wish to introduce a regulation scheme such that we can factor $\Pi^{ab}_{\mu\nu}(k)$ as

$$\Pi^{ab}_{\mu\nu}(k) = \delta^{ab}(k_\mu k_\nu - g_{\mu\nu}k^2)\,\Pi(k^2) \tag{7.18}$$

where $\Pi(k^2)$ only diverges logarithmically.

From (7.18), we can write the full propagator in the Lorentz gauge as (Fig. 7.2)

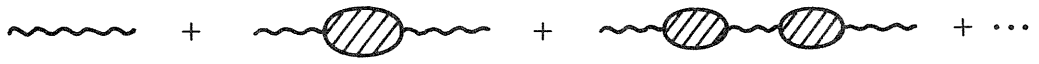


Fig. 7.2 The full gauge-field propagator.

$$iD^{ab}_{\mu\nu}(k) = \frac{-i\delta^{ab}(g_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2 + i\varepsilon} + \frac{-i\,\delta^{aa'}(g_{\mu\mu'} - k_\mu k_{\mu'}/k^2)}{k^2 + i\varepsilon}$$

$$\times (i\delta^{a'b'}(k_{\mu'}k_{\nu'} - g_{\mu'\nu'}k^2)\,\Pi(k^2)) \times \frac{-i\,\delta^{b'b}(g_{\nu'\nu} - k_{\nu'}k_\nu/k^2)}{k^2 + i\varepsilon} + \cdots$$

$$= \frac{-i\delta^{ab}(g_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2(1 + \Pi(k^2))}. \tag{7.19}$$

Thus, if (7.18) is valid, we can renormalize (7.19) readily.

In order to achieve (7.18), we have to introduce a regulator scheme which is consistent with the gauge invariance. The Pauli-Villars method is no longer working for a non-abelian gauge theory because the cutoff method destroys the gauge invariance. We need a different method.

In 1972, 't Hooft and Veltman introduced a regularization procedure known as the dimension regularization. The idea works as follows: The Yang-Mills theory (or any non-abelian gauge theory) is invariant in any space-time dimension n. In evaluating a loop integral, we only need to replace the d^4q integration by

$$\Pi^{ab}_{\mu\nu} = \int \frac{d^nq}{(2\pi)^n}\,(\text{same integrand}). \tag{7.20}$$

We shall concentrate on the ultraviolet divergence. For small enough n ($n < 2$), all loop-integrations converge. We can indeed put $\Pi^{ab}_{\mu\nu}(k)$ into the form of (7.18). Then, we renormalize $\Pi(k^2)$ by writing

$$\Pi(k^2) = \Pi(0) + \Pi_R(k^2) \tag{7.21}$$

with

$$\Pi(0) = Z_3^{-1} - 1. \tag{7.22}$$

We then let $n \to 4$ in $\Pi_R(k^2)$ at the end of the calculation.

To see how the method works in practice, we compute the QED vacuum polarization diagram which is given in Fig. 7.3.

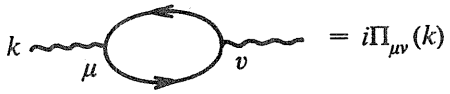


Fig. 7.3 Second-order QED vacuum polarization diagram.

The QED vacuum polarization amplitude in n dimensions ($n < 2$) is

$$\begin{aligned} i\Pi_{\mu\nu}(k) &= -e^2 \int \frac{d^n q}{(2\pi)^n} \frac{\mathrm{Tr}\,[\gamma_\mu(\not{q}-\not{k}+m)\,\gamma_\nu(\not{q}+m)]}{[(q-k)^2-m^2+i\varepsilon][q^2-m^2+i\varepsilon]} \\ &= -e^2 \int_0^1 dx \int \frac{d^n q}{(2\pi)^n} \frac{\mathrm{Tr}\,[\gamma_\mu(\not{q}-\not{k}+m)\,\gamma_\nu(\not{q}+m)]}{[(q-xk)^2+x(1-x)k^2-m^2+i\varepsilon]^2} \\ &= -e^2 \int_0^1 dx \int \frac{d^n q'}{(2\pi)^n} \frac{\mathrm{Tr}\,[\gamma_\mu(\not{q}'-(1-x)\not{k}+m)\,\gamma_\nu(\not{q}'+x\not{k}+m)]}{[q'^2+x(1-x)k^2-m^2+i\varepsilon]^2}. \end{aligned} \tag{7.23}$$

Now, we look at the numerator of (7.23). We can ignore terms linear in q'. The remaining terms are

$$N_{\mu\nu} = 4\,[2q'_\mu q'_\nu - g_{\mu\nu}q'^2 - x(1-x)\,(2k_\mu k_\nu - g_{\mu\nu}k^2) + m^2 g_{\mu\nu}]. \tag{7.24}$$

Under the integration, we can also replace $q'_\mu q'_\nu$ in $N_{\mu\nu}$ by $g_{\mu\nu}\,q'^2/n$, obtaining

$$N_{\mu\nu} \to 4\,[2g_{\mu\nu}\,q'^2/n - g_{\mu\nu}q'^2 - x(1-x)\,(2k_\mu k_\nu - g_{\mu\nu}k^2) + m^2\,g_{\mu\nu}]. \tag{7.25}$$

We encounter the integrations of the form

$$\int d^n q \frac{1}{(q^2-A)^\alpha} = (-1)^\alpha\, i \left[\int d^n q \frac{1}{(q^2+A)^\alpha}\right]_{\text{Euclidean}}$$

$$= (-1)^\alpha i \int_0^\infty \frac{2\pi^{n/2}}{\Gamma(n/2)} q^{n-1} dq \frac{1}{(q^2+A)^\alpha}$$

$$= i(-1)^\alpha \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \frac{d(q^2)\,(q^2)^{\frac{n}{2}-1}}{(q^2+A)^\alpha}$$

$$= i(-1)^\alpha \frac{\pi^{n/2}}{\Gamma(n/2)} A^{\frac{n}{2}-\alpha} \frac{\Gamma(n/2)\,\Gamma(\alpha-n/2)}{\Gamma(\alpha)}$$

$$= i(-1)^\alpha \pi^{n/2} A^{\frac{n}{2}-\alpha} \Gamma(\alpha - n/2)/\Gamma(\alpha), \quad (7.26)$$

$$\int \frac{d^n q\, q^2}{(q^2-A)^\alpha} = i(-1)^{\alpha-1} \pi^{n/2} A^{\frac{n}{2}+1-\alpha} \frac{n}{2} \Gamma\left(\alpha-\frac{n}{2}-1\right)/\Gamma(\alpha). \quad (7.27)$$

Hence, we have

$$i\Pi_{\mu\nu}(k) = -e^2 \int_0^1 dx \frac{4}{(2\pi)^n} \left\{ \left(\frac{2}{n}-1\right) g_{\mu\nu} \frac{(-i)\,\pi^{n/2} \frac{n}{2} \Gamma(1-\frac{n}{2})/\Gamma(2)}{(-x(1-x)\,k^2+m^2)^{1-n/2}} \right.$$

$$\left. + \frac{(-x(1-x)\,(2k_\mu k_\nu - g_{\mu\nu} k^2) + m^2 g_{\mu\nu}) i\pi^{n/2} \Gamma(2-\frac{n}{2})/\Gamma(2)}{(-x(1-x)\,k^2+m^2)^{2-n/2}} \right.$$

$$= \frac{4ie^2}{(2\pi)^n} \int_0^1 dx \frac{\Gamma\left(2-\frac{n}{2}\right)\pi^{n/2}}{\Gamma(2)}$$

$$\times \left\{ \frac{g_{\mu\nu}(-x(1-x)k^2+m^2)+x(1-x)(2k_\mu k_\nu - g_{\mu\nu}k^2) - m^2 g_{\mu\nu}}{(-x(1-x)\,k^2+m^2)^{2-n/2}} \right\}$$

$$= \frac{i4e^2\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right)}{(2\pi)^n\,\Gamma(2)} \int_0^1 dx \frac{2x(1-x)\,(k_\mu k_\nu - g_{\mu\nu}k^2)}{(-x(1-x)\,k^2+m^2)^{2-n/2}}. \quad (7.28)$$

Thus, we can write

$$\Pi_{\mu\nu}(k) = (k_\mu k_\nu - g_{\mu\nu}k^2)\,\Pi(k^2) \tag{7.29}$$

with

$$\Pi(k^2) = \frac{4e^2\pi^{n/2}\,\Gamma(2-n/2)}{(2\pi)^n\Gamma(2)} \int_0^1 \frac{dx\; 2x(1-x)}{(-x(1-x)\,k^2+m^2)^{2-n/2}}\,. \tag{7.30}$$

Without the $\left(\frac{2}{n}-1\right)$ factor, expression $\Pi_{\mu\nu}$ would have divergence at $n=2$ which corresponds to a quadratic divergence at four dimensions. Expression (7.30) is finite for $n < 4$, and diverges as $1/(n-4)$ at $n \to 4$ which corresponds to a logarithmic divergence at $n=4$. To obtain the renormalized Π_R, we subtract the integrand of (7.30) at $k=0$,

$$\begin{aligned}\Pi_R(k^2) &= \Pi(k^2) - \Pi(0)\\ &= \frac{4e^2\pi^{n/2}\,\Gamma(2-n/2)}{(2\pi)^n\Gamma(2)} \int_0^1 dx\; 2x(1-x)\\ &\quad \times \left\{\frac{1}{(-x(1-x)\,k^2+m^2)^{2-n/2}} - \frac{1}{(m^2)^{2-n/2}}\right\}.\end{aligned} \tag{7.31}$$

Expression (7.31) is now finite as $n \to 4$, giving

$$\begin{aligned}\Pi_R(k^2) &= \lim_{n\to 4} \frac{4e^2\,\pi^{n/2}\,\Gamma(2-n/2)}{(2\pi)^n\,\Gamma(2)} \int_0^1 dx\; 2x(1-x)\\ &\quad \times \{-(2-n/2)\ln[(-x(1-x)\,k^2+m^2)/m^2]\}\\ &= \lim_{n\to 4} \frac{4e^2\pi^{n/2}\,\Gamma(3-n/2)}{(2\pi)^n\,\Gamma(2)}\\ &\quad \times \left(-\int_0^1 dx\; 2x(1-x)\ln\frac{-x(1-x)\,k^2+m^2}{m^2}\right)\end{aligned}$$

$$= -\frac{4e^2\pi^2}{(2\pi)^4}\int_0^1 dx\, 2x(1-x)\ln\left(1 - x(1-x)\,k^2/m^2\right)$$

$$= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left(1 - x(1-x)\,k^2/m^2\right). \qquad (7.32)$$

This is the same result that we have obtained earlier by the Pauli-Villars method.

Next, we wish to identify the renormalization constant

$$Z_3^{-1} - 1 = \Pi(0)$$

$$= \lim_{n\to 4}\frac{4e^2}{(2\pi)^n}\frac{\pi^{n/2}\,\Gamma(2-n/2)}{\Gamma(2)}\int_0^1 \frac{dx\, 2x(1-x)}{(m^2)^{2-n/2}}$$

$$= \lim_{n\to 4}\frac{1}{2-n/2}\frac{4e^2\pi^{n/2}\,\Gamma(3-n/2)}{(2\pi)^n\,\Gamma(2)}\frac{(m^2)^{n/2-2}}{3}$$

$$= \frac{e^2}{6\pi^2}\frac{1}{4-n} + \text{finite terms at } n = 4$$

$$= \frac{2\alpha}{3\pi}\frac{1}{4-n} + O(1). \qquad (7.33)$$

We compare (7.33) with our previous expression

$$Z_3^{-1} - 1 = \frac{\alpha}{3\pi}\ln M^2/m^2 + O(1)$$

which leads to the identification

$$\frac{1}{4-n} \leftrightarrow \ln M/m \qquad (7.34)$$

where M is the ultraviolet cutoff.

We can now use the dimension regularization method to study non-abelian gauge theories. Consider graphs 7.1a–d. Using the gauge propagator

$$D^{ab}_{F\mu\nu}(k) = \frac{-i\,(g_{\mu\nu} - (1-\beta)\,k_\mu k_\nu/k^2)}{k^2+i\varepsilon}, \tag{7.35}$$

we can indeed express the pure gauge field self-energy diagrams (Fig. 7.1a–c), under the dimension regularization, as

$$\Pi^{(g)ab}_{\mu\nu}(k) = \delta^{ab}(k_\mu k_\nu - g_{\mu\nu}k^2)\Pi^{(g)}(k^2). \tag{7.36}$$

We shall not work out the detailed algebra, but quote only the final result:

$$\begin{aligned}\Pi^{(g)}(k^2) = {} & \frac{g^2}{(2\pi)^n}\frac{\pi^{n/2}}{(-k^2)^{2-n/2}}C_1 \\ & \times\left\{2\left[B\left(\frac{n}{2},\frac{n}{2}\right) - B\left(\frac{n}{2}-1,\frac{n}{2}-1\right)\right]\Gamma(2-n/2)\right. \\ & + (1-\beta)\left[2\,B\left(\frac{n}{2}-1,\frac{n}{2}\right)\Gamma\left(3-\frac{n}{2}\right)\right. \\ & \left. - B\left(\frac{n}{2}-1,\frac{n}{2}\right)\Gamma\left(2-\frac{n}{2}\right)\right] \\ & \left. - (1-\beta)^2\frac{1}{4}\Gamma\left(3-\frac{n}{2}\right)B\left(\frac{n}{2}-1,\frac{n}{2}-1\right)\right\}\end{aligned} \tag{7.37}$$

where $C_1 = 2$ for Yang-Mills fields. Since we have no scales other than k^2, the factor $g^2/(k^2)^{2-n/2}$ in $\Pi(k^2)$ is expected by dimension analysis. However, the expression in the curl brackets can only come from explicit calculations.

It is important to note that the ghost term is needed for gauge invariance and for obtaining the $(k_\mu k_\nu - g_{\mu\nu}k^2)$ factor in (7.36).

As mentioned earlier, the fermion loop contribution (Fig. 7.1d) is essentially the same as in QED,

$$\Pi^{(f)ab}_{\mu\nu} = \delta^{ab}\,(k_\mu k_\nu - g_{\mu\nu}k^2)\,\Pi^{(f)}\,(k^2) \tag{7.38}$$

with

$$\Pi^{(f)}(k^2) = \frac{4g^2}{(2\pi)^n} \frac{\pi^{n/2}\,\Gamma(2-n/2)}{\Gamma(2)} C_2$$

$$\times \int_0^1 \frac{dx\, 2x(1-x)}{(-x(1-x)\,k^2+m^2)^{2-n/2}}. \tag{7.39}$$

From (7.37) and (7.39), we can work out the finite part Π_R and Z_3. In particular, the logarithmic divergent part in Z_3 is the pole term $1/(4-n)$ in $\Pi(k^2)$,

$$Z_3^{-1} - 1 = -\frac{g^2 C_1}{16\pi^2}\left(\frac{5}{3} + \frac{1-\beta}{2}\right)\frac{2}{4-n}$$

$$+ \frac{g^2 C_2}{12\pi^2}\frac{2}{4-n}. \tag{7.40}$$

Unlike the QED result, Z_3 here is gauge dependent. The parameters C_1, C_2 are Casimir operators related to the adjoint representation of the gauge group, and to the fermion representation T^a,

$$f^{acd} f^{bcd} = C_1 \delta^{ab} \tag{7.41a}$$

$$\mathrm{Tr}\,(T^a T^b) = C_2 \delta^{ab}. \tag{7.41b}$$

3. Other Renormalization Parts

There are many different kinds of renormalization parts in a non-abelian gauge theory.

In Fig. 7.4, we plot all one-loop contributions to the three-gauge-field vertex. These diagrams all diverge logarithmically, and can be made

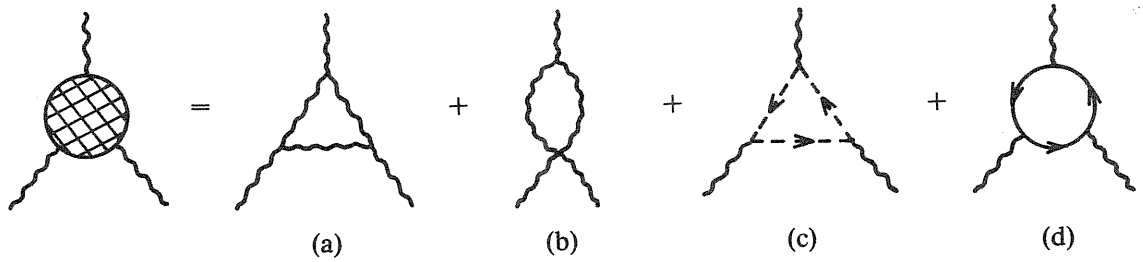


Fig. 7.4 One-loop contributions to the three-gauge-field vertex.

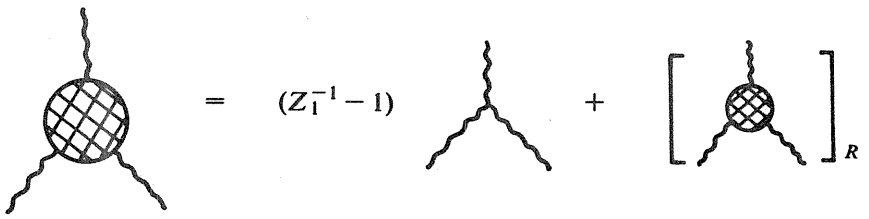


Fig. 7.5 The separation of the three-gauge-field vertex contribution into an infinite bare vertex and a finite part.

finite by one subtraction as shown in Fig. 7.5. We identify the renormalization constant as Z_1, and denote the finite part by a subscript R.

The four-gauge-field vertex also diverges logarithmically which leads to a new renormalization constant Z_4. In Fig. 7.6, we plot all one-loop

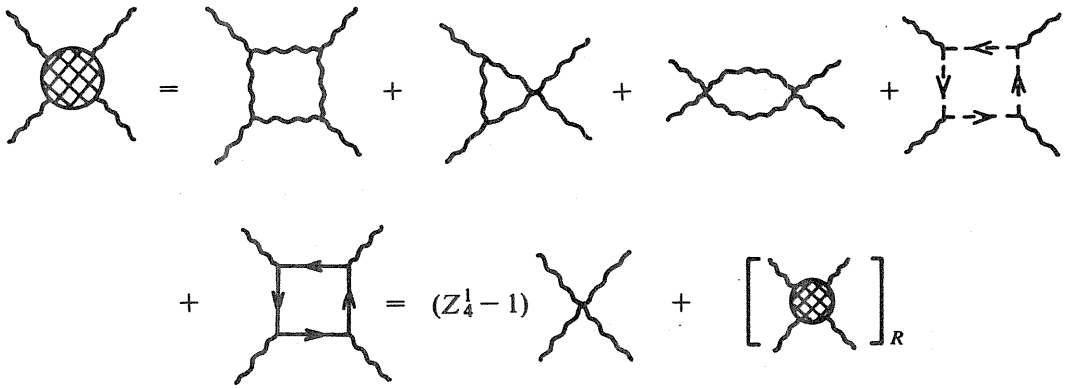


Fig. 7.6 One-loop contributions to and subtraction procedure for a four-gauge-field vertex.

contributions and the subtraction procedure. Since the classical four-gauge-field vertex has a coupling constant g^2 which is the same as the square of three-gauge-vertex, we expect that a similar result may hold for the properly renormalized quantities. In particular, this would imply that Z_4 is not independent, and can be expressed in terms of Z_1 and Z_3.

Even though the ghosts do not appear as external particles, they also contribute to renormalization procedure. In Fig. 7.7a, b, we show the ghost self-energies and vertices. They lead to two more renormalization constants $\tilde{Z}_3$ and $\tilde{Z}_1$. Since the ghost in fact represents the determinant of the gauge field, we expect that $\tilde{Z}_1$, $\tilde{Z}_3$ are related to Z_1 and Z_3. Note also that the ghosts do not couple directly to fermions.

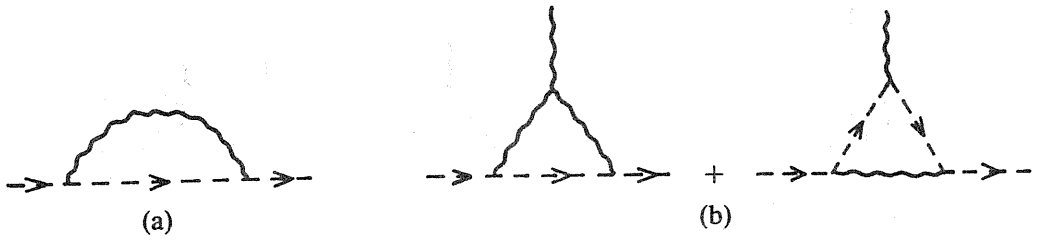


Fig. 7.7 (a) One-loop ghost self-energy diagram, (b) ghost-gauge-field vertices.

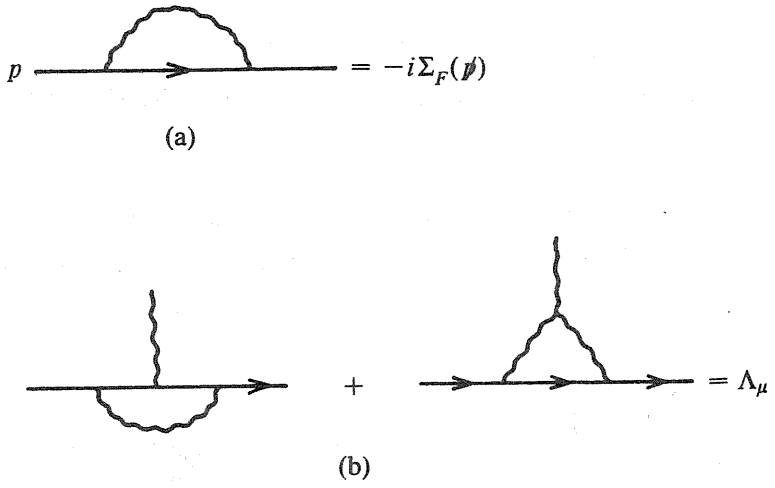


Fig. 7.8 (a) One-loop fermion self-energy, (b) gauge-field fermion vertices.

Finally, we have the fermion self-energies and vertices as shown in Fig. 7.8. We can renormalize these graphs as in QED, and lead to a mass renormalization δm, and two renormalization constants Z_2 and Z_{1F}. These conclude all the renormalization parts.

In the following, we shall establish several relations between the renormalized coupling constant g and the unrenormalized coupling constant g_0. Note that the unrenormalized coupling constant g_0 for the 3-gauge-field vertex, gague-field-ghost vertex, and the gauge-field-fermion vertex are the same. The 4-gauge-field vertex is g^2. If the gauge symmetry survives the renormalization, we would expect that, with a proper choice of renormalization procedure, the renormalized coupling constant g obtained from the renormalized 3-gauge-field vertex, the renormalized gauge-field-ghost vertex, and the renormalized gauge-field-fermion vertex are also the same. Similarly, we expect the g^2 obtained from the renormalized 4-gauge-field vertex is the square of the same g. One can show that these relations are indeed valid to one-loop level.

To relate the renormalized coupling to the unrenormalized coupling constant, we follow the same procedure as described in QED. To obtain

g, we multiply g_0 by the Z^{-1} associated with the vertex, and the $\sqrt{Z}$ associated with each of the legs. For a 3-gauge-field vertex, we have

$$g = Z_1^{-1} Z_3^{3/2} g_0. \tag{7.42}$$

The gauge-field-fermion vertex gives

$$g = Z_{1F}^{-1} Z_2 Z_3^{1/2} g_0. \tag{7.43}$$

and the gauge-field-ghost vertex gives

$$g = \tilde{Z}_1^{-1} \tilde{Z}_3 Z_3^{1/2} g_0. \tag{7.44}$$

The 4-gauge-field vertex gives

$$g^2 = Z_4^{-1} Z_3^2 g_0^2. \tag{7.45}$$

From (7.42)–(7.45), we find the consistency requirements,

$$\frac{Z_4}{Z_1} = \frac{Z_1}{Z_3} = \frac{\tilde{Z}_1}{\tilde{Z}_3} = \frac{Z_{1F}}{Z_2}. \tag{7.46}$$

These relations are the generalization of QED results $Z_1 = Z_2$, and guarantee that the renormalized coupling constants g are the same for the above four kinds of vertices.

4. Slavnov-Taylor Identities

Slavnov and Taylor obtained the generalized Ward-Takahashi identities in the non-abelian gauge theories. To understand their derivation, we begin with the generating functional for QED Green's functions (See Chap. 5, Eq. (7.47))

$$W\{j, \eta, \bar{\eta}\} = \int \mathscr{D}\{A\}\, \mathscr{D}\{\psi\}\, \mathscr{D}\{\bar{\psi}\}$$

$$\times \exp\left\{ i \int d^4x \left[\mathscr{L}(x) - \frac{1}{2\beta} (\partial_\mu A_\mu)^2 - A_\mu j_\mu + \bar{\eta}\psi + \bar{\psi}\eta \right] \right\}. \tag{7.47}$$

We consider an infinitesimal gauge transformation

$$A_\mu \to A_\mu(x) + \partial_\mu \delta\phi(x), \tag{7.48a}$$

$$\psi(x) \to e^{ie\delta\phi(x)}\,\psi(x) = \psi(x) + ie\,\delta\phi(x)\,\psi(x), \tag{7.48b}$$

$$\bar{\psi}(x) \to \bar{\psi}(x)\,e^{-ie\delta\phi(x)} = \bar{\psi}(x) - ie\,\delta\phi\,\bar{\psi}(x). \tag{7.48c}$$

Since this is a transformation on the (path) integration variables, $W\{j, \eta, \bar{\eta}\}$ is invariant under such a transformation. Note also that the QED Lagrange function $\mathscr{L}(x)$ is invariant under this gauge transformation. However, the gauge fixing term and the terms coupled to j, η and $\bar{\eta}$ are not invariant under this transformation. Hence, under such a gauge transformation, we have

$$\begin{aligned}
\delta W &= 0 \\
&= \int \mathscr{D}\{A\}\,\mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\} \exp\left\{ i \int d^4x' \right. \\
&\quad \times \left. \left[\mathscr{L}(x') - \frac{1}{2\beta}(\partial_\mu A_\mu)^2 - A_\mu j_\mu + \bar{\eta}\psi + \bar{\psi}\eta \right] \right\} \\
&\quad \times i \int d^4x \left[-\frac{1}{\beta}(\partial_\mu A_\mu)\,\partial^2\delta\phi - j_\mu \partial_\mu \delta\phi \right. \\
&\quad \left. + \bar{\eta}\,\psi(x)\,ie\delta\phi - \bar{\psi}\eta ie\delta\phi \right].
\end{aligned} \tag{7.49}$$

Let

$$\delta\chi \equiv \partial^2\delta\phi \tag{7.50}$$

be the independent gauge parameter. We have

$$\begin{aligned}
&\int \mathscr{D}\{A\}\,\mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\} \exp\left\{ i \int d^4x' \,[\mathscr{L}(x') + \cdots] \right\} \\
&\times \int d^4x \left(-\frac{1}{\beta}\partial_\mu A_\mu + (\partial^2)^{-1}\,\partial_\mu j_\mu + ie\,(\partial^2)^{-1}\,\bar{\eta}\psi \right. \\
&\left. - ie(\partial^2)^{-1}\bar{\psi}\eta \right) \delta\chi(x) = 0.
\end{aligned} \tag{7.51}$$

Since (7.51) is valid for all gauge parameter $\delta\phi$ (or $\delta\chi$), we have

$$\int \mathscr{D}\{A\}\,\mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\}\left[-\frac{1}{\beta}\partial_\mu A_\mu + (\partial^2)^{-1}\,\partial_\mu j_\mu + ie\,(\partial^2)^{-1}\,\bar{\eta}\psi - ie\,(\partial^2)^{-1}\,\bar{\psi}\eta\right]$$

$$\times \exp\left\{i\int d^4x'\,[\mathscr{L}(x') + \cdots]\right\} = 0. \tag{7.52}$$

Finally, we express A_μ, ψ, and $\bar{\psi}$ in terms of functional differentiation operators, and obtain

$$\left[-\frac{1}{\beta}\partial_\mu\left(-\frac{1}{i}\frac{\delta}{\delta j_\mu}(x)\right) + (\partial^2)^{-1}\,\partial_\mu j_\mu + ie\,(\partial^2)^{-1}\,\bar{\eta}\,\frac{1}{i}\frac{\delta}{\delta\bar{\eta}}\right.$$

$$\left. - ie\,(\partial^2)^{-1}\,\eta\,\frac{1}{i}\frac{\delta}{\delta\eta}\right] W\{j, \eta, \bar{\eta}\} = 0. \tag{7.53}$$

Equation (7.53) is a compact expression of the Ward-Takahashi identities. Expanding (7.53) to order η and $\bar{\eta}$, we obtain the usual Ward identity.

Now, we consider the pure Yang-Mills theory. The generating functional for the Green's functions in the Gaussian gauge is

$$W\{j\} = \int \mathscr{D}\{\mathbf{B}\}\,\det\{A\}$$

$$\times \exp\left\{i\int d^4x\left[\mathscr{L}(x) - \frac{1}{2\beta}(\partial_\mu B^a_\mu(x))^2 - \mathbf{B}_\mu\cdot\mathbf{j}_\mu\right]\right\} \tag{7.54}$$

where

$$\mathscr{L}(x) = -\frac{1}{4}G^{a2}_{\mu\nu},$$

$$A^{ab} \equiv \delta^{ab}\partial^2 - g\,\varepsilon^{abc}B^c_\mu\partial_\mu - g\,\varepsilon^{abc}\partial_\mu B^c_\mu\,. \tag{7.55}$$

In (7.54), $\mathscr{L}$ is the Lagrange function for the gauge invariant non-abelian field theory. Under a gauge transformation, $\mathscr{D}\{\mathbf{B}\}\,\det\{A\}$ is invariant. Hence, in our derivation, we do not need to introduce the ghost fields.

We consider an infinitesimal gauge transformation

$$\delta \mathbf{B}_\mu = g\delta\boldsymbol{\omega} \times \mathbf{B}_\mu - \partial_\mu \delta\boldsymbol{\omega}. \tag{7.56}$$

The Lagrange function $\mathscr{L}$ and the det $\{A\}$ are invariant under (7.56), but the gauge fixing term and the $B_\mu j_\mu$ term are not. Equation (7.56) leads to the change of action S,

$$S \equiv \int d^4x \left(\mathscr{L} - \frac{1}{2\beta} (\partial_\mu \mathbf{B}_\mu)^2 - \mathbf{B}_\mu \cdot \mathbf{j}_\mu \right), \tag{7.57}$$

$$\delta S = \int d^4x \left[-\frac{1}{\beta} (\partial_\nu \mathbf{B}_\nu) \cdot \partial_\mu (g\,\delta\boldsymbol{\omega} \times \mathbf{B}_\mu - \partial_\mu \delta\boldsymbol{\omega}) \right.$$

$$\left. - \mathbf{j}_\mu \cdot (g\delta\boldsymbol{\omega} \times \mathbf{B}_\mu - \partial_\mu \delta\boldsymbol{\omega}) \right]. \tag{7.58}$$

We define $-\partial_\mu \delta \mathbf{B}_\mu$ as $\delta\boldsymbol{\chi}$,

$$\begin{aligned} -\partial_\mu \delta B_\mu^a &= -\partial_\mu (g\delta\boldsymbol{\omega} \times \mathbf{B}_\mu - \partial_\mu \delta\boldsymbol{\omega})^a \\ &= \partial^2 \delta\omega^a - g(\partial_\mu \delta\boldsymbol{\omega} \times \mathbf{B}_\mu)^a - g(\delta\boldsymbol{\omega} \times \partial_\mu \mathbf{B}_\mu)^a \\ &= A^{ab} \delta\omega^b \\ &\equiv \delta\chi^a. \end{aligned} \tag{7.59}$$

Hence, we can invert (7.59) to give

$$\delta\omega = A^{-1}\,\delta\chi \tag{7.60}$$

where A^{-1} is the "ghost" Green's function in the presence of $\mathbf{B}_\mu$, and obeys

$$(\delta^{ab}\partial^2 - g\varepsilon^{abd} B_\mu^d \partial_\mu - g\varepsilon^{abd} \partial_\mu B_\mu^d)(A^{-1}(x, y))^{bc} = \delta^{ac}\delta(x - y). \tag{7.61}$$

Then, we have

$$\delta S = \int d^4x \left[\frac{1}{\beta} \partial_\mu B_\mu^d + g\,\varepsilon^{abc} j_\mu^a B_\mu^b (A^{-1})^{cd} \right.$$

$$\left. - \partial_\mu j_\mu^a (A^{-1})^{ad} \right] \delta\chi^d. \tag{7.62}$$

Since W is invariant under this change of variables, we have

$$\delta W = 0$$

$$= \int \mathscr{D}\{\mathbf{B}\} \det\{A\} \exp\left\{ i \int d^4x' \, \mathscr{L}(x') + \cdots \right\}$$

$$\times \int d^4x \left[\frac{1}{\beta} \partial_\mu B_\mu^d - D_\mu j_\mu^a (A^{-1})^{ad} \right] \delta\chi^d$$

which implies

$$\int \mathscr{D}\{[\mathbf{B}]\} \det\{[A]\} \exp\left\{ i \int d^4x' \mathscr{L}(x') + \cdots \right\}$$

$$\times \left[\frac{1}{\beta} \partial_\mu B_\mu^d - (D_\mu j_\mu)^a (A^{-1})^{ad} \right] = 0. \tag{7.63}$$

We can rewrite (7.63) as

$$\left[\frac{1}{\beta} \partial_\mu \left(-\frac{1}{i} \frac{\delta}{\delta j_\mu^d} \right) - \left(\partial_\mu j_\mu^a - g\varepsilon^{abc} j_\mu^b \left(-\frac{1}{i} \frac{\delta}{\delta j_\mu^c} \right) \right) \right.$$

$$\left. \times A^{-1} \left(i \frac{\delta}{\delta j} \right)^{ad} \right\} W\{j\} = 0. \tag{7.64}$$

Equations (7.61) and (7.64) are the functional equation form of the Slavnov-Taylor identities. These identities are the generalization of Ward-Takahashi identities to non-abelian gauge theories. From the Slavnov-Taylor identities, we can establish the desired relations among Z's as expressed in Sec. 3.

Since the ghost propagator $A^{-1}(\mathbf{B})$ is a nonpolynomial function of $\mathbf{B}$, its presence in (7.64) gives rise to a nonlocal functional differential equation and greatly complicates the Slavnov-Taylor identities. One way to eliminate the ghost propagator in (7.64) is to introduce the ghost fields ϕ, ϕ^* and their sources ξ, ξ^* explicitly.

$$W\{j, \xi, \xi^*\} = \int \mathscr{D}\{\mathbf{B}\} \int \mathscr{D}\{\phi\} \, \mathscr{D}\{\phi^*\}$$

$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{4}\mathbf{G}_{\mu\nu}^2 - \frac{1}{2\beta}(\partial_\mu \mathbf{B}_\mu)^2 + \phi^{*a}\left(\delta^{ab}\partial^2 - g\varepsilon^{abc}B_\mu^c\partial_\mu \right.\right.\right.$$

$$\left.\left.\left. - g\,\varepsilon^{abc}\,\partial_\mu B_\mu^c\right)\phi^b - B_\mu^a j_\mu^a + \xi^*\phi + \phi^*\xi \right]\right\} \tag{7.65a}$$

$$= \int \mathscr{D}\{\mathbf{B}\}\det\{A(\mathbf{B})\}$$

$$\times \exp\left\{ i \int d^4x \left[-\frac{1}{4}\mathbf{G}_{\mu\nu}^2 - \frac{1}{2\beta}(\partial_\mu \mathbf{B}_\mu)^2 - B_\mu^a j_\mu^a \right.\right.$$

$$\left.\left. - \xi^{a*}A^{-1}(B)^{ab}\xi^b \right]\right\}. \tag{7.65b}$$

It is easy to see that $W\{j\}$ is related to $W\{j, \xi, \xi^*\}$ at $\xi = \xi^* = 0$,

$$W\{j\} = W\{j, \xi, \xi^*\}\big|_{\xi=\xi^*=0}. \tag{7.66}$$

In terms of $W\{j, \xi, \xi^*\}$, Eq. (7.64) becomes

$$\left[\frac{1}{\beta}\partial_\mu\left(-\frac{1}{i}\frac{\delta}{\delta j_\mu^d}\right) - \left(\partial_\mu j_\mu^a - g\varepsilon^{abc}j_\mu^b\left(-\frac{1}{i}\frac{\delta}{\delta j_\mu^c}\right)\right)\right]$$

$$\times (i)\frac{1}{i}\frac{\delta}{\delta\xi^{*a}}\frac{1}{i}\frac{\delta}{\delta\xi^d}\Bigg] W\{j, \xi, \xi^*\} = 0. \tag{7.67}$$

By paying the price of introducing the ghost fields and currents, we obtain a finite-order functional differential equation, (7.67). We can recover Eq. (7.64) from (7.67) trivially by setting $\xi = \xi^* = 0$.

5. Higgs Mechanism

We have pointed out that a naive inclusion of mass terms to gauge fields will make the theory non-renormalizable. Higgs discovered that if a gauge invariant theory develops masses for its gauge fields through a spontaneous symmetry-breaking, then, it preserves its renormalizability. We shall illustrate this kind of spontaneous mass generation by an example.

Consider a classical Maxwell field coupled to a classical charged scalar field ϕ,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 + (D_\mu\phi)^\dagger D_\mu\phi + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{7.68}$$

where

$$D_\mu\phi(x) \equiv (\partial_\mu - ieA_\mu)\,\phi(x). \tag{7.69}$$

Note that we have chosen a wrong sign of the ϕ (mass)^2 term. The potential due to the ϕ term is

$$V(|\phi|) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2. \tag{7.70}$$

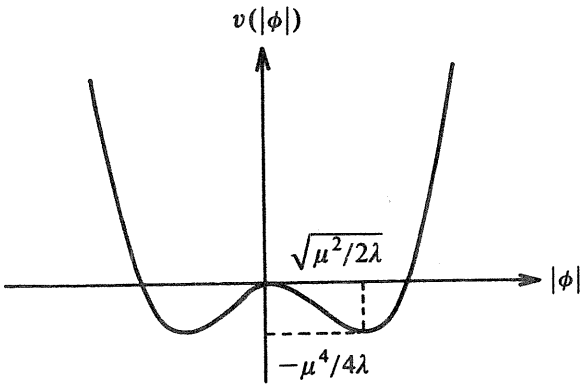


Fig. 7.9 The potential energy density as a function of $|\phi|$.

The minimum of V is at $V'=0$, giving

$$\phi^\dagger\phi \equiv |\phi|^2 = \mu^2/2\lambda$$

or

$$|\phi| = \pm\sqrt{\mu^2/2\lambda}. \tag{7.71}$$

Thus, the ground state is specified by (7.71). It is easy to verify that both the Lagrange function $\mathscr{L}$ and the ground state condition (7.68) are invariant under the gauge transformation

$$\phi(x) \to \phi'(x) = e^{-ie\lambda(x)}\,\phi(x) \tag{7.72a}$$

$$\phi^\dagger(x) \to \phi^{\dagger\prime}(x) = \phi^\dagger(x)\,e^{-ie\lambda(x)} \tag{7.72b}$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\lambda(x). \tag{7.73}$$

Using the gauge degree of freedom, we can choose $(\phi)_{\text{ground state}}$ to be real. In terms of two real field ϕ_1 and ϕ_2 defined by

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \tag{7.74a}$$

$$\phi^\dagger = \frac{\phi_1 - i\phi_2}{\sqrt{2}}, \tag{7.74b}$$

we can choose

$$\phi_1 = \sqrt{\mu^2/\lambda} \equiv v \tag{7.75}$$

$$\phi_2 = 0. \tag{7.76}$$

We can generalize Eqs. (7.75)–(7.76) quantum-mechanically to

$$\langle \phi_1 \rangle = v, \quad \langle \phi_2 \rangle = 0. \tag{7.77}$$

Expanding $\mathscr{L}$ around (7.74),

$$\phi_1 = v + \phi_1', \qquad \phi_2 = 0 + \phi_2', \tag{7.78}$$

we have

$$\begin{aligned}
\mathscr{L} &= -\frac{1}{4}F_{\mu\nu}^2 + |(\partial_\mu - ieA_\mu)\phi|^2 + \mu^2|\phi|^2 - \lambda|\phi^2|^2 \\
&= -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}[(\partial_\mu\phi_1 + eA_\mu\phi_2)^2 + (\partial_\mu\phi_2 - eA_\mu\phi_1)^2] \\
&\quad + \mu^2|\phi|^2 - \lambda|\phi^2|^2 \\
&= -\frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2}[(\partial_\mu\phi_1' + eA_\mu\phi_2')^2 + (\partial_\mu\phi_2' - eA_\mu\phi_1')^2] \\
&\quad - evA_\mu(\partial_\mu\phi_2' - eA_\mu\phi_1') + \frac{1}{2}e^2v^2A_\mu^2 \\
&\quad + \frac{\mu^4}{4\lambda} - \lambda v^2\phi_1'^2 - \lambda v(\phi_1'^2 + \phi_2'^2)\phi_1' - \frac{\lambda}{4}(\phi_1'^2 + \phi_2'^2)^2.
\end{aligned} \tag{7.79}$$

In (7.79), the constant term $\mu^4/4\lambda$ describes the energy density of the new ground state. The terms linear in ϕ' drops out by the requirement $V'(\phi)=0$. The new (mass)^2 for ϕ_1' is now positive. The ϕ_2' (the phase in the original ϕ) and A_μ term combine to give rise to a massive vector field

$$
\begin{aligned}
&-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - evA_\mu\partial_\mu\phi_2' + \frac{1}{2}(\partial_\mu \phi'_2)^2 + \frac{1}{2}e^2v^2A_\mu^2 \\
&=-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2}e^2v^2\left(A_\mu - \frac{1}{ev}\partial_\mu\phi_2'\right)^2 \\
&=-\frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}m^2B_\mu^2, \qquad (7.80)
\end{aligned}
$$

where

$$
B_\mu \equiv A_\mu - \frac{1}{ev}\partial_\mu\phi_2' \qquad (7.81)
$$

and

$$
m = ev. \qquad (7.82)
$$

Without the gauge field, ϕ_2' would be the Goldstone mode. In (7.80), the massless gauge field A_μ combines with the gradient of this would-be Goldstone boson ϕ_2' to become a massive vector field. The above mechanism is called the Higgs phenomenon. We can extend the above theory to a quantum field theory, and obtain a renormalizable massive gauge theory.

We can generalize the Higgs mechanism to a non-abelian gauge theory. The Weinberg-Salam theory of electroweak interaction is based on such a generalization. 't Hooft, B. Lee, Zinn-Justin, etc, have shown that the resultant theories are indeed renormalizable.

Chapter 8

RENORMALIZATION GROUP

The renormalization group is a relatively old technique in quantum field theory. In recent years, Callan and Symanzik reduced the renormalization group transformation into a differential equation which becomes very effective in studying the asymptotic behavior of Green's functions. Wilson applied the renormalization group method to the field of critical phenomena which has revolutionized our understanding of phase transitions.

1. Intermediate Renormalization and its Group Property

We consider a ϕ^4 field theory

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!}g\phi^4. \tag{8.1}$$

This theory is renormalizable in $3+1$ dimension. We need a mass and a wave function renormalization for the self-energy of ϕ, a vertex renormalization for the 4-ϕ vertex.

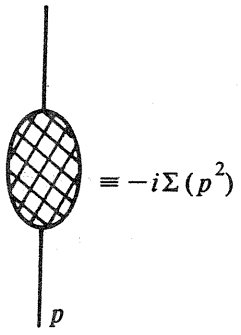


Fig. 8.1 The proper self-energy diagram.

The standard method of constructing the renormalized self-energy part is to expand $\Sigma(p^2)$ in Fig. 8.1 around $p^2 = \mu^2$, giving

$$\Sigma(p^2) = \Sigma(\mu^2) + \Sigma'(\mu^2)(p^2-\mu^2) + \tilde{\Sigma}(p^2) \tag{8.2}$$

where $\tilde{\Sigma}(p^2) = O((p^2-\mu^2)^2)$ as $p^2 \to \mu^2$. The unrenormalized and the renormalized propagator are

$$i\Delta'(p^2) = \frac{i}{p^2-\mu^2-(\Sigma(p^2)-\delta\mu^2)} = Z_2 i\tilde{\Delta}(p^2), \tag{8.3}$$

$$i\tilde{\Delta}(p^2) = \frac{i}{p^2-\mu^2-\tilde{\Sigma}(p^2)} \tag{8.4}$$

where

$$\Sigma(\mu^2) = \mu^2 - \mu_0^2 \equiv \delta\mu^2 \tag{8.5}$$

and

$$\Sigma'(\mu^2) = 1 - Z_2^{-1}. \tag{8.6}$$

The normalization conditions on the inverse-renormalized propagator,

$$(i\tilde{\Delta}(p^2))^{-1} = -i(p^2-\mu^2-\tilde{\Sigma}(p^2)) \equiv \tilde{\Gamma}^{(2)}(p^2) \tag{8.7}$$

are

$$\tilde{\Delta}(p^2)^{-1}\big|_{p^2=\mu^2} = 0 \tag{8.8}$$

and

$$\frac{\partial}{\partial p^2}\tilde{\Delta}(p^2)^{-1}\big|_{p^2=\mu^2} = 1. \tag{8.9}$$

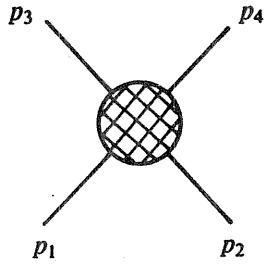


Fig. 8.2 The 4-point vertex.

For the 4-point Green's function, we denote the unrenormalized Green's function by $\Gamma'^{(4)}$ and the renormalized Green's function by $\tilde{\Gamma}^{(4)}$ with*

$$\Gamma'^{(4)}(\{p\}) = Z_2^{-2}\,\tilde{\Gamma}^{(4)}(\{p\}). \tag{8.10}$$

One of the standard renormalization points is

$$\tilde{\Gamma}^{(4)}(\{p\}) = g \tag{8.11}$$

where $\{p\}$ is chosen at the symmetric point

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = \mu^2,$$

$$s = t = u = 4\mu^2/3. \tag{8.12}$$

In Eq. (8.12), s, t, u are the Mandelstein variables

$$s \equiv (p_1+p_2)^2,\ t \equiv (p_1+p_3)^2,\ u \equiv (p_1+p_4)^2. \tag{8.13}$$

However, the renormalization point chosen above is just for convenience. We may choose any other momentum for renormalization. For instance, we may choose an arbitrary scale ξ^2 and define the normalization conditions as

$$\left.\frac{\partial\tilde{\Gamma}^{(2)}(p^2)}{\partial p^2}\right|_{p^2=\xi^2} = -i \tag{8.14}$$

$$\tilde{\Gamma}^{(2)}(\xi^2) = -i(\xi^2 - \mu_\xi^2) \tag{8.15}$$

and

$$\tilde{\Gamma}^{(4)}(\{p\})_{\text{s.p.}} = -ig_\xi \tag{8.16}$$

*If we normalize $\tilde{\Gamma}^{(4)}$ by $\tilde{\Gamma}^{(4)}(\{p\}) = 1$ at the renormalization point. Eq. (8.10) should be modified to $g_0\,\Gamma'^{(4)}(\{p\}) = Z_2^{-2}\,g\,\tilde{\Gamma}^{(4)}(\{p\})$ which follows from $\Gamma'^{(4)}(\{p\}) = Z_4^{-1}\tilde{\Gamma}^{(4)}(\{p\})$ and $g = Z_2^2 Z_4^{-1} g_0$.

where $\tilde{\Gamma}^{(4)}$ is defined at the symmetric point of $p_i^2 = \xi^2$. If we choose μ_ξ and g_ξ appropriately, the two theories lead to the same physical amplitudes. Actually, there is no reason for choosing the symmetric point, we can choose any fixed momenta as the renormalization points.

We consider two equivalent renormalizations, one at ξ_1, μ_1 and g_1, and another at ξ_2, μ_2 and g_2. The unrenormalized Green's function and the renormalized Green's function are related by

$$\begin{aligned}\Gamma^{(2)}(p, \Lambda, \mu_0, g_0) &= Z_2(\xi_1, \Lambda, \mu_0, g_0)^{-1}\, \tilde{\Gamma}^{(2)}(p, \xi_1, \mu_1, g_1) \\ &= Z_2(\xi_2, \Lambda, \mu_0, g_0)^{-1}\, \tilde{\Gamma}^{(2)}(p, \xi_2, \mu_2, g_2), \end{aligned} \tag{8.17}$$

$$\begin{aligned}\Gamma^{(4)}(\{p\}, \Lambda, \mu_0, g_0) &= Z_2(\xi_1, \Lambda, \mu_0, g_0)^{-2}\, \tilde{\Gamma}^{(4)}(\{p\}, \xi_1, \mu_1, g_1) \\ &= Z_2(\xi_2, \Lambda, \mu_0, g_0)^{-2}\, \tilde{\Gamma}^{(4)}(\{p\}, \xi_2, \mu_2, g_2), \end{aligned} \tag{8.18}$$

where Λ is an ultraviolet cutoff, μ_0 and g_0 are bare mass and coupling constant. From (8.17) and (6.18), we have

$$\tilde{\Gamma}^{(2)}(p, \xi_2, \mu_2, g_2) = (z(\xi_2, \xi_1, \mu_1, g_1))^{-1}\, \tilde{\Gamma}^{(2)}(p, \xi_1, \mu_1, g_1) \tag{8.19}$$

$$\tilde{\Gamma}^{(4)}(\{p\}, \xi_2, \mu_2, g_2) = [z(\xi_2, \xi_1, \mu_1, g_1)]^{-2}\, \tilde{\Gamma}^{(4)}(\{p\}, \xi_1, \mu_1, g_1). \tag{8.20}$$

In other words, given ξ_1, μ_1 and z_1, we can compute $\tilde{\Gamma}^{(n)}(\{p\}, \xi_2, \mu_2, g_2)$ for a different renormalization point. We can also compute g_2, μ_2 in terms of ξ_2, ξ_1, μ_1, g_1

$$g_2 = g(\xi_2, \xi_1, \mu_1, g_1) \tag{8.21}$$

$$\mu_2 = \mu(\xi_2, \xi_1, \mu_1, g_1). \tag{8.22}$$

We may repeat the procedure to find g_3, μ_3 for a third renormalization point. We may either compute them directly from ξ_1, μ_1, g_1, or indirectly from ξ_2, μ_2, g_2. These two procedures should lead to the same result

$$\begin{aligned} g_3 &= g(\xi_3, \xi_2(\xi_1, \mu_1, g_1), \mu_2(\xi_1, \mu_1, g_1), g_2(\xi_1, \mu_1, g_1)) \\ &= g(\xi_3, \xi_1, \mu_1, g_1), \end{aligned} \tag{8.23}$$

$$\begin{aligned} \mu_3 &= \mu(\xi_3, \xi_2(\xi_1, \mu_1, g_1), \mu_2(\xi_1, \mu_1, g_1), g_2(\xi_1, \mu_1, g_1)) \\ &= \mu(\xi_2, \xi_1, \mu_1, g_1). \end{aligned} \tag{8.24}$$

Obviously, Eqs. (8.19), (8.20), (8.23) and (8.24) have the transformation properties of a group. This transformation group is called the renormalization group. The method described above is the old-fashion formulation of the renormalization group, and it is very awkward to extract physical consequences from it.

2. Callan-Symanzik Equation

Consider an unrenormalized n-point Green's function $\Gamma^{(n)}(\{p\}, \Lambda, \mu_0, g_0)$ and the corresponding renormalized Green's function $\tilde{\Gamma}^{(n)}(\{p\}, \xi, \mu, g)$. They are related by

$$\Gamma^{(n)}(\{p\}, \Lambda, \mu_0, g_0) = Z_2^{-n/2}\,\tilde{\Gamma}^{(n)}(\{p\}, \xi, \mu,(\xi, \cdots), g(\xi, \cdots)). \tag{8.25}$$

Obviously, the unrenormalized Green's function does not know about the renormalization point. This is Callan and Symanzik's starting point of the renormalization group formulation. Differentiating $\Gamma^{(n)}$ with respect to ξ, we have

$$\begin{aligned} \xi\frac{\partial}{\partial\xi}(\Gamma^{(n)}(\{p\}, \Lambda, \mu_0, g_0)) &= \xi\frac{\partial}{\partial\xi}[Z_2^{-n/2}\,\tilde{\Gamma}^{(n)}(\{p\}, \xi, \mu, g)] \\ &= 0. \end{aligned} \tag{8.26}$$

In (8.26), we have kept the unrenormalized quatities Λ, μ_0 and g_0, fixed. Under such a condition, we may consider μ and g as functions of ξ. We then have

$$\begin{aligned} & Z_2^{-n/2}(-n/2)\left(\xi\frac{\partial}{\partial\xi}Z_2\right)\tilde{\Gamma}^{(n)}/Z_2 \\ & + Z_2^{-n/2}\left[\xi\frac{\partial}{\partial\xi}\tilde{\Gamma}^{(n)}(\{p\}, \xi, \mu, g) + \left(\xi\frac{\partial}{\partial\xi}g(\xi)\right)\frac{\partial}{\partial g}\tilde{\Gamma}^{(n)}(\{p\}, \xi, \mu, g)\right. \\ & \left. + \left(\xi\frac{\partial}{\partial\xi}\mu(\xi)\right)\frac{\partial}{\partial\mu}\tilde{\Gamma}^{(n)}(\{p\}, \xi, \mu, g)\right] \\ & = 0. \end{aligned} \tag{8.27}$$

Introducing the notation

$$\beta = \xi \frac{\partial}{\partial \xi} g(\xi) \tag{8.28a}$$

$$\gamma = \frac{1}{2} Z_2^{-1} \xi \frac{\partial}{\partial \xi} Z_2 \tag{8.28b}$$

$$\xi \frac{\partial \mu}{\partial \xi} \frac{\partial}{\partial \mu} \tilde{\Gamma}^{(n)} \equiv - \tilde{\Gamma}_{\Delta}^{(n)}, \tag{8.29}$$

we have

$$\left[\xi \frac{\partial}{\partial \xi} + \beta(g) \frac{\partial}{\partial g} - n\gamma(g) \right] \tilde{\Gamma}^{(n)}(\{p\}, \xi, \mu, g)$$

$$= \tilde{\Gamma}_{\Delta}^{(n)}(\{p\}, \xi, \mu, g). \tag{8.30}$$

Equation (8.30) is the Callan-Symanzik relation. $\tilde{\Gamma}_{\Delta}^{(n)}$ is related to a mass insertion term. We can interpret (8.30) as the differential equation form of the renormalization equation about $\tilde{\Gamma}^{(n)}$.

At very large momenta $\{p\}$, the mass insertion term $\tilde{\Gamma}_{\Delta}$ and the μ dependence are negligible. We then have the asymptotic equation

$$\left(\xi \frac{\partial}{\partial \xi} + \beta \frac{\partial}{\partial g} - n\gamma \right) \tilde{\Gamma}^{(n)}(\{p\}, \xi, g) = 0. \tag{8.31}$$

Note that $g(\xi)$, $\mu(\xi)$, and ξ are the analogs of g_0, μ_0, and the ultraviolet cutoff Λ. We can also derive an equivalent Callan-Symanzik relation in terms of $\mu \frac{\partial}{\partial \mu}$.

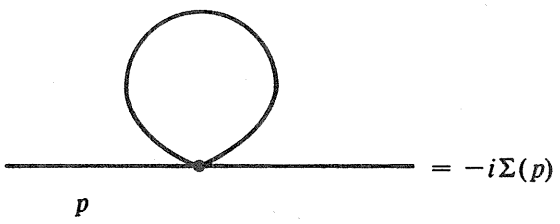


Fig. 8.3 The lowest-order self-energy diagram in the ϕ^4 theory.

We shall work out the lowest order $\beta(g)$ and $\gamma(g)$ in the ϕ^4 theory. The lowest order self-energy diagram is Fig. 8.3. To order g, $\Sigma(p)$ is independent of p. Thus, we have

$$1 - Z_2^{-1} = \Sigma'(\xi^2) = 0.$$

Hence, we have

$$\gamma = O(g^2) \qquad (\phi^4 \text{ theory}). \tag{8.32}$$

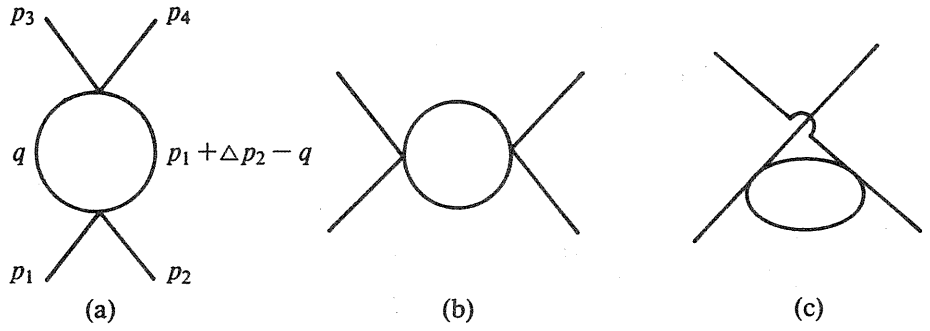


Fig. 8.4 The lowest-order 4-point vertex contributions.

To order g^2, the contribution of $g(\xi)$ arises from diagrams (a)–(c) in Fig. 8.4. For Fig. 8.4a, we have

$$\Lambda^{(a)} = \frac{1}{2}(ig_0)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2-\mu_0^2+i\varepsilon} \frac{i}{(p_1+p_2-q)^2-\mu_0^2+i\varepsilon} . \tag{8.33}$$

It is easy to work out the logarithmic divergent part of (8.33) as

$$\Lambda^{(a)}(\{p\}, \cdots) = \frac{ig_0^2}{32\pi^2}\left(\ln \Lambda^2/\xi^2 + \text{finite part}\right). \tag{8.34}$$

Adding contributions from Figs. 8.4a–c, we have

$$\Lambda(\{p\}, \cdots) = \frac{3ig_0^2}{32\pi^2}\left(\ln \Lambda^2/\xi^2 + \text{finite part}\right). \tag{8.35}$$

Recall that $Z_2 = 1$. Thus, the renormalized coupling constant, $g(\xi)$, is

$$g = g_0 - \frac{3g_0^2}{32\pi^2} \ln \Lambda^2/\xi^2 + O(g_0^3)$$

$$= g_0 \left(1 + \frac{3g_0}{32\pi^2} \ln \Lambda^2/\xi^2\right)^{-1} + O(g_0^3). \tag{8.36}$$

In (8.36), we really have no compelling reason to write it as a fraction.

However, the expression is the analog of Z_4^{-1}, and is a reasonable choice. When we differentiate (8.36), we have

$$\beta(g) = \xi \frac{\partial g}{\partial \xi} = \frac{\dfrac{3g_0^2}{32\pi^2} \cdot 2}{\left(1 + \dfrac{3g_0^2}{32\pi^2} \ln \Lambda^2/\xi^2\right)^2}$$

$$= \frac{3g^2}{16\pi^2} + O(g^3) \qquad (\phi^4 \text{ theory}). \tag{8.37}$$

Note that the fractional form reproduces itself. As we shall see in the next section, this particular functional form appears naturally in a renormalization group calculation.

Let us also work out the lowest order expression of β for quantum electrodynamics (QED). We recall our earlier result

$$\alpha = \frac{\alpha_0}{1 + \dfrac{\alpha_0}{3\pi} \ln \Lambda^2/\xi^2} \,. \tag{8.38}$$

Defining $\beta(\alpha)$ as

$$\beta(\alpha) \equiv \xi \frac{\partial}{\partial \xi} \alpha(\xi), \tag{8.39}$$

we have

$$\beta(\alpha) = \frac{\dfrac{\alpha_0^2}{3\pi} \cdot 2}{\left(1 + \dfrac{\alpha_0}{3\pi} \ln \Lambda^2/\xi^2\right)^2}$$

$$= \frac{2\alpha^2}{3\pi} + O(\alpha^3) \qquad (\text{QED}). \tag{8.40}$$

As we have seen, we can compute β and γ in perturbation calculation straightforwardly.

3. Solutions to Callan-Symanzik Equations

We shall solve the asymptotic Callan-Symanzik equation

$$\left[\xi\frac{\partial}{\partial\xi}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)\right]\tilde{\Gamma}^{(n)}(\{p\},\xi,g)=0. \tag{8.31}$$

We consider the limit of $\mu/\xi=0$, $\mu/p=0$, and with p's going to ∞ at the same rate. We can factor out a naive (canonical) p factor,

$$\tilde{\Gamma}^{(n)}(\{p\},\xi,g)=(p^2)^{\text{naive factor}}\rho^{(n)}(p/\xi,\{p_i/p_j\},g) \tag{8.41}$$

and obtain

$$\left[\xi\frac{\partial}{\partial\xi}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)\right]\rho^{(n)}(p/\xi,\cdots,g)=0. \tag{8.42}$$

where p is a typical momentum. We introduce a new variable t,

$$t\equiv\ln p/\xi.$$

Then, we have

$$\left[-\frac{\partial}{\partial t}+\beta(g)\frac{\partial}{\partial g}-n\gamma(g)\right]\rho^{(n)}(t,\cdots,g)=0. \tag{8.43}$$

As pointed out by Coleman, Eq. (8.43) is of the form

$$\frac{\partial\rho}{\partial t}+v(x)\frac{\partial\rho}{\partial x}=L(x)\rho \tag{8.44}$$

which has a hydrodynamics analog. We may interpret (8.44) as the change of population of bacteria with density ρ moving along a pipe with the fluid. Here, t describes the time, and x describes the distance along the pipe. The fluid has a known position-dependent velocity $v(x)$ along the pipe. As the bacteria move along, they are subject to an x-dependent illumination which determines their reproduction rate $L(x)$.

To find $\rho(x,t)$, we can follow a column of bacteria along the pipe. At $t=0$, we select a column of bacteria at position x. At time t, the fluid is located at $x'(x,t)$. Then, we have

$$\frac{dx'(x,t)}{dt} = v(x'), \tag{8.45}$$

under the initial condition

$$x'(x,0) = x. \tag{8.46}$$

We can solve Eqs. (8.46) and (8.47) trivially to obtain $x'(x,t)$. The time translational invariance tells us that $x'(x, t_1 - t_2)$ describes the position of the fluid at t_1 if it is at x at t_2. Given that the bacteria density at $t=0$ is $f(x)$, we can write down the bacteria density at time t as

$$\rho(x,t) = f(x'(x,-t)) \exp\left[\int_0^t dt'\, L(x'(x,-t'))\right]$$

$$= f(x'(x,-t)) \exp\left[\int_{-t}^0 dt'\, L(x'(x,t'))\right]. \tag{8.47}$$

In (8.48), $x'(x,-t)$ describes the location of the bacteria at $t=0$ when it is at location x at time t. Equation (8.48) is the required solution.

We shall use this analog to find the solution of (8.44). We have the following identifications

$$g \leftrightarrow x \tag{8.48a}$$

$$\beta(g) \leftrightarrow -v(x) \tag{8.48b}$$

$$n\gamma(g) \leftrightarrow -L(x). \tag{8.48c}$$

We introduce an "invariant coupling constant" $\bar{g}$ through

$$\frac{d\bar{g}(g,t)}{dt} = -\beta(\bar{g}) \tag{8.49}$$

with

$$\bar{g}(g,0) = g. \tag{8.50}$$

Then, we have

$$\rho^{(n)}(\{p\}, \xi, g) = \rho^{(n)}(\{p^0\}, \xi, \bar{g}(g, -t))$$

$$\times \exp\left[-\int_{-t}^{0} dt' \, n\gamma(\bar{g}(g, t'))\right]. \tag{8.51}$$

Knowing $\rho^{(n)}$ for a finite momentum ρ^0 and all $\bar{g}$, we can compute $\rho^{(n)}$ and $\tilde{\Gamma}^{(n)}$ for all momentum $p = e^t p^0$.

Historically, the invariant charge $\bar{g}(t)$ is defined by

$$\frac{d\bar{g}(t)}{dt} = \beta(\bar{g})$$

and

$$\bar{g}(0) = g.$$

This invariant charge $g(t)$ is related to our $\bar{g}(g, t)$ by

$$\bar{g}(t) = \bar{g}(g, -t). \tag{8.52}$$

$\bar{g}(t)$ is also known as a running coupling constant.

To get some feeling about $\bar{g}$, we work out the invariant charge associated with the lowest order ϕ^4 β-function,

$$\frac{d\bar{g}(g, t)}{dt} = -\beta(\bar{g}) = -\frac{3\bar{g}^2}{16\pi^2}, \tag{8.53}$$

with

$$\bar{g}(g, 0) = g. \tag{8.54}$$

Hence, we have

$$\frac{1}{\bar{g}} = \frac{3t}{16\pi^2} + \text{const}$$

$$= \frac{3t}{16\pi^2} + \frac{1}{g}. \tag{8.55}$$

We obtain

$$\bar{g}(g, t) = \frac{g}{1 + \frac{3g}{16\pi^2}t}$$

$$= \frac{g}{1 + \frac{3g}{16\pi^2}\ln p/\xi}. \tag{8.56}$$

The running coupling constant is

$$\bar{g}(t) = \bar{g}(g, -t)$$

$$= \frac{g}{1 - \frac{3g}{16\pi^2}t}$$

$$= \frac{g}{1 - \frac{3g}{16\pi^2}\ln p/\xi}. \tag{8.57}$$

It is important to see that $\bar{g}(t)$ increases as p increases. In other words, the running coupling constant becomes stronger at shorter distance. The same behavior also occurs in QED.

4. An Example

We shall use the second order QED photon self-energy diagram,

$$D'_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2[1 + \Pi(q^2, \Lambda, \alpha_0)]} \tag{8.58}$$

$$\Pi(q^2, \Lambda, \alpha_0) = \frac{\alpha_0}{3\pi}\ln \Lambda^2/q^2, \quad (q^2 >> m^2) \tag{8.59}$$

as an example to illustrate the renormalization group technique. In the following, we shall make no further approximations, and treat (8.59) and (8.60) as if they were exact.

We introduce a renormalization point ξ and expand Π around $q^2 = \xi^2$, giving

$$\Pi(q^2, \Lambda, \alpha_0) = \frac{\alpha_0}{3\pi} \ln \Lambda^2/\xi^2 + \frac{\alpha_0}{3\pi} \ln \xi^2/q^2. \tag{8.60}$$

We make the identifications

$$Z_3^{-1} - 1 = \frac{\alpha_0}{3\pi} \ln \Lambda^2/\xi^2,$$

or

$$Z_3(\xi, \Lambda, \alpha_0)^{-1} = 1 + \frac{\alpha_0}{3\pi} \ln \Lambda^2/\xi^2, \tag{8.61}$$

and

$$Z_3^{-1} \tilde{\Pi}(q^2, \xi, \alpha) = \frac{\alpha_0}{3\pi} \ln \xi^2/q^2. \tag{8.62}$$

The renormalized charge e and α are

$$e = \sqrt{Z_3}\, e_0, \tag{8.63a}$$

$$\alpha = Z_3 \alpha_0 = \frac{\alpha_0}{1 + \frac{\alpha_0}{3\pi} \ln \Lambda^2/\xi^2}. \tag{8.63b}$$

In terms of the renormalized α, we have

$$\tilde{\Pi}(q^2, \xi, \alpha) = \frac{\alpha}{3\pi} \ln \xi^2/q^2 \tag{8.64}$$

and

$$D'_{\mu\nu}(q^2) = \frac{-ig_{\mu\nu} Z_3(\xi, \Lambda, \alpha_0)}{q^2 [1 + \tilde{\Pi}(q^2, \xi, \alpha)]}$$

$$= \frac{-ig_{\mu\nu} Z_3(\xi, \Lambda, \alpha_0)}{q^2 \left(1 + \frac{\alpha}{3\pi} \ln \xi^2/q^2\right)}. \tag{8.65}$$

It is natural to introduce a renormalized photon propagator as

$$\tilde{D}_{\mu\nu}(q, \xi, \alpha) \equiv Z_3^{-1} D'_{\mu\nu}$$

$$= \frac{-ig_{\mu\nu}}{q^2\left[1 + \frac{\alpha}{3\pi} \ln \xi^2/q^2\right]}. \tag{8.66}$$

If we do the calculation around two different ξ's, we have

$$D'_{\mu\nu}(q, \Lambda, \alpha_0) = \frac{-ig_{\mu\nu} Z_3(\xi_1, \Lambda, \alpha_0)}{q^2[1 + \tilde{\Pi}(q^2, \xi_1, \alpha_1)]}$$

$$= \frac{-ig_{\mu\nu} Z_3(\xi_2, \Lambda, \alpha_0)}{q^2[1 + \tilde{\Pi}(q^2, \xi_2, \alpha_2)]}. \tag{8.67}$$

Hence, we have

$$\tilde{D}_{\mu\nu}(q, \xi_2, \alpha_2) = \frac{-ig_{\mu\nu}}{q^2[1 + \tilde{\Pi}(q^2, \xi_2, \alpha_2)]}$$

$$= Z_3(\xi_2, \xi_1, \alpha_1^{-1})^{-1} \tilde{D}_{\mu\nu}(q, \xi_1, \alpha_1) \tag{8.68}$$

with

$$z_3(\xi_2, \xi_1, \alpha_1)^{-1} \equiv \frac{Z_3(\xi_1, \Lambda, \alpha_0)}{Z_3(\xi_2, \Lambda, \alpha_0)}$$

$$= \frac{1 + \frac{\alpha_0}{3\pi} \ln \Lambda^2/\xi_2^2}{1 + \frac{\alpha_0}{3\pi} \ln \Lambda^2/\xi_1^2}$$

$$= 1 + \frac{\frac{\alpha_0}{3\pi} \ln \xi_1^2/\xi_2^2}{1 + \frac{\alpha_0}{3\pi} \ln \Lambda^2/\xi_1^2}$$

$$= 1 + \frac{\alpha_1}{3\pi} \ln \xi_1^2/\xi_2^2 \tag{8.69}$$

which is finite. We can also show that

$$\begin{aligned}\alpha_2 &= Z_3(\xi_2, \Lambda, \alpha_0)\,\alpha_0 \\ &= \frac{Z_3(\xi_2, \Lambda, \alpha_0)}{Z_3(\xi_1, \Lambda, \alpha_0)}\,\alpha_1 \\ &= z_3(\xi_2, \xi_1, \alpha_1)\,\alpha_1 \\ &= \frac{\alpha_1}{1 + \dfrac{\alpha_1}{3\pi}\ln \xi_1^2/\xi_2^2}.\end{aligned} \tag{8.70}$$

It is straightforward to verify that the transformation laws (8.69)–(8.71) form a group.

What is the large q behavior of $\tilde{D}_{\mu\nu}(q)$? Naively, we would expect

$$\lim_{q^2\to\infty} \tilde{D}_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2} \qquad \text{(false)}. \tag{8.71}$$

Since

$$z_3(\xi_2, \xi_1, \alpha_1) \neq 1,$$

we know that (8.72) cannot be valid for both ξ_1 and ξ_2. In fact (8.72) is false for all ξ. The correct expression is

$$\lim_{q^2\to\infty} \tilde{D}_{\mu\nu}(q) = \frac{-ig_{\mu\nu}}{q^2}\,\rho(q/\xi, \alpha) \tag{8.72}$$

as given by (8.67).

We now follow the Callan-Symanzik approach to study this system. From (8.40), we have

$$\beta(\alpha) = \frac{2}{3\pi}\alpha^2. \tag{8.40'}$$

The $\gamma(\alpha)$ is

$$\gamma \equiv \frac{1}{2} Z_3^{-1} \xi \frac{\partial}{\partial \xi} Z_3$$

$$= \frac{\alpha}{3\pi}. \tag{8.73}$$

$\tilde{\Gamma}^{(2)}$ is the inverse propagator. Ignoring the $-ig_{\mu\nu}$ factor, we have

$$\tilde{\Gamma}^{(2)} = q^2 [1 + \Pi(q^2, \xi, \alpha)]. \tag{8.74}$$

The Callan-Symanzik equation implies that $\tilde{\Gamma}^{(2)}$ obeys

$$\left(\xi \frac{\partial}{\partial \xi} + \frac{2}{3\pi} \alpha^2 \frac{\partial}{\partial \alpha} - 2 \cdot \frac{\alpha}{3\pi}\right) \tilde{\Gamma}^{(2)}(q^2, \xi, \alpha) = 0. \tag{8.75}$$

We first check whether our $\tilde{\Gamma}^{(2)}$ in (8.75) obeys (8.76). From (8.67), we have

$$\tilde{\Gamma}^{(2)} = q^2 \left(1 + \frac{\alpha}{3\pi} \ln \xi^2/q^2\right).$$

Then, we obtain

$$\text{lhs of (8.76)} = q^2 \left[\frac{2\alpha}{3\pi} + \frac{2\alpha}{3\pi} \frac{1}{3\pi} \ln \xi^2/q^2 - \frac{2\alpha}{3\pi} \left(1 + \frac{\alpha}{3\pi} \ln \xi^2/q^2\right)\right]$$

$$= 0$$

as desired.

Next, using the method outlined in the previous section, we wish to show that we can reproduce our $\tilde{D}_{\mu\nu}$ by solving (8.75). We obtain the invariant charge in terms of α and $t = \ln p/\xi$. The invariant charge $\bar{\alpha}$ obeys

$$\frac{d\bar{\alpha}(\alpha, t)}{dt} = -\beta(\bar{\alpha}) = -\frac{2}{3\pi} \bar{\alpha}^2 \tag{8.76a}$$

$$\bar{\alpha}(\alpha, 0) = \alpha \tag{8.76b}$$

giving

$$\frac{1}{\bar{\alpha}(\alpha, t)} = \frac{1}{\alpha} + \frac{2}{3\pi} t. \tag{8.77}$$

Hence, we have

$$\bar{\alpha}(\alpha, t) = \frac{\alpha}{1 + \frac{2\alpha}{3\pi} t}. \tag{8.78}$$

The asymptotic behavior of $\tilde{\Gamma}^{(2)}$ is

$$\tilde{\Gamma}^{(2)} = q^2 \rho^{(2)} \tag{8.79}$$

with

$$\begin{aligned}
\rho^{(2)}(q^2, \xi, \alpha) &= \rho^{(2)}(q_0^2, \xi, \bar{\alpha}(\alpha, -t)) \exp\left[- \int_{-t}^{0} dt' \frac{2}{3\pi} \bar{\alpha}(\alpha, t') \right] \\
&= \rho^{(2)}(q_0^2, \xi, \bar{\alpha}(\alpha, -t)) \exp\left[- \ln\left(1 + \frac{2\alpha}{3\pi} t'\right) \Big|_{-t}^{0} \right] \\
&= \rho^{(2)}(q_0^2, \xi, \bar{\alpha}(\alpha, -t)) \left(1 + \frac{2\alpha}{3\pi}(-t)\right).
\end{aligned} \tag{8.80}$$

At the renormalization point, $q_0^2 = \xi^2 (t=0), \rho^{(2)} = 1$ by construction. We have

$$\begin{aligned}
\rho^{(2)}(q^2, \xi, \alpha) &= 1 + \frac{2\alpha}{3\pi}(-t) \\
&= 1 + \frac{2\alpha}{3\pi} \ln \xi^2/q^2
\end{aligned} \tag{8.81}$$

$$\tilde{\Gamma}^{A(2)}(q^2, \xi, \alpha) = q^2 \left(1 + \frac{2\alpha}{3\pi} \ln \xi^2/q^2\right) \tag{8.82}$$

as desired. We have thus determined the asymptotic behavior of $\tilde{D}_{\mu\nu} = -ig_{\mu\nu}/\tilde{\Gamma}^{(2)}$ by its renormalization group property.

5. Asymptotically Free Theories

We have discussed in the previous chapter how to compute the renormalization constants Z's and the renormalization parts for a purely

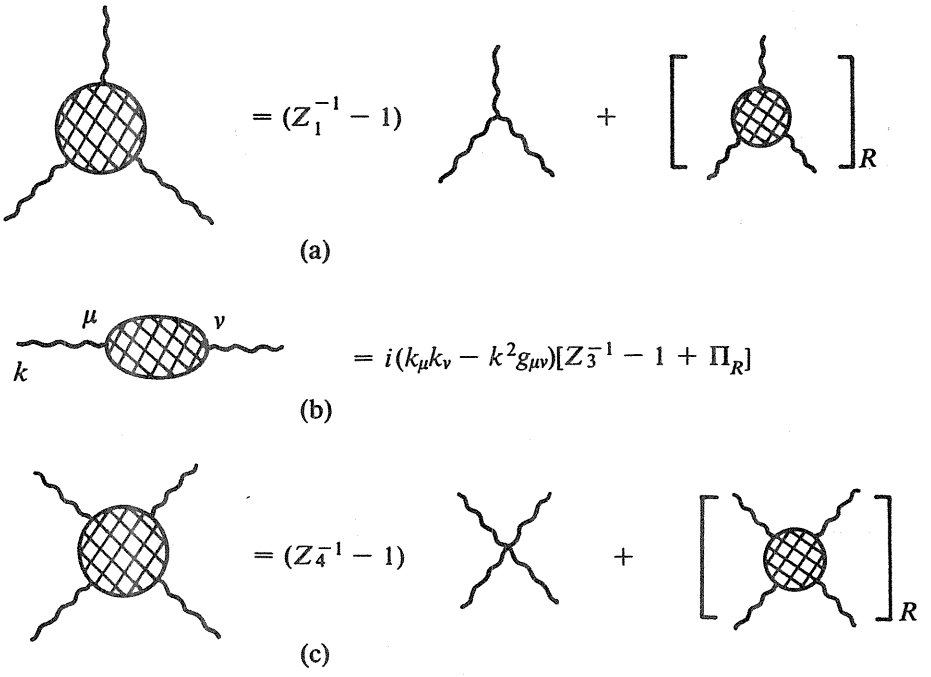


Fig. 8.5 The renormalization parts in a non-abelian gauge theory.

non-abelian gauge theory. The Z_1, Z_3 and Z_4 are defined symbolically in Fig. 8.5. We summarize the one-loop results as follows

$$Z_1 = 1 + \frac{g^2 C_1}{16\pi^2}\left[\frac{2}{3} + \frac{3}{4}(1-\beta)\right] \ln \Lambda^2/\xi^2 \tag{8.83a}$$

$$Z_3 = 1 + \frac{g^2 C_1}{16\pi^2}\left[\frac{5}{3} + \frac{1}{2}(1-\beta)\right] \ln \Lambda^2/\xi^2 \tag{8.83b}$$

$$Z_4 = 1 + \frac{g^2 C_1}{16\pi^2}\left[-\frac{1}{3} + (1-\beta)\right] \ln \Lambda^2/\xi^2 \tag{8.83c}$$

where C_1 is the Casimir operator for the adjoint representation,

$$C_1 = N \text{ for group SU}(N). \tag{8.84}$$

It is pleasing to see that Eq. (8.84) obeys the identities, to $O(g^2)$,

$$\frac{Z_4}{Z_1} = \frac{Z_1}{Z_3}. \tag{8.85}$$

The renormalized charge g is related to g_0 via

$$g = Z_3^{3/2} Z_1^{-1} g_0$$

$$= Z_3 Z_4^{-1/2} g_0$$

$$= \left(1 + \frac{g^2 C_1}{16\pi^2} \times \frac{11}{6} \ln \Lambda^2/\xi^2\right) g_0 + O(g^5). \qquad \text{(Purely gauge field)} \tag{8.86}$$

In the presence of fermion fields, Eq. (8.87) is modified to

$$g = \left[1 + \frac{g^2}{16\pi^2}\left(\frac{11C_1}{6} - \frac{2}{3}C_2\right) \ln \Lambda^2/\xi^2\right] g_0 \tag{8.87}$$

where C_2 is the Casimir operator associated with the fermion representation,

$$\mathrm{Tr}\,(T^a T^b) = \delta^{ab} C_2\,. \tag{8.88}$$

For the N-dimensional basic representation of SU(N) gauge group, $C_2 = 1/2$.

The β for the non-abelian gauge field with fermions is

$$\beta = \xi \frac{\partial}{\partial \xi} g(\xi, \Lambda, g_0)$$

$$= -\frac{g^3}{16\pi^3}\left(\frac{11C_1}{3} - \frac{4C_2}{3}\right) + O(g^5). \tag{8.89}$$

The coefficient of g^3 is negative if $11C_1 > 4C_2$. For QCD, $C_1 = 3$ (group SU(3)), we can have 16 triplets of fermions in the fundamental representation without affecting the sign of β.

We now study the running constant at small g when β is of the form

$$\beta(g) = -ag^3, \qquad a > 0. \tag{8.90}$$

We have the equation for the invariant charge as

$$\frac{d}{dt}\bar{g}(g, t) = -\beta(\bar{g}) = a\bar{g}^3 \tag{8.91}$$

with

$$\bar{g}(g, 0) = g. \tag{8.92}$$

Solving (8.92) and (8.93), we have

$$\frac{1}{\bar{g}^2} = \frac{1}{g^2} - 2at, \tag{8.93}$$

or

$$\bar{g}(g, t)^2 = \frac{g^2}{1 - 2ag^2t}. \tag{8.94}$$

The running coupling constant is

$$\bar{g}(t) = \bar{g}(g, -t)$$

$$= \frac{g}{\sqrt{1 + 2ag^2t}}. \tag{8.95}$$

Thus, as $t = \ln p/\xi \to \infty$, the running coupling constant goes to zero. Such a theory is known as asymptotically free. The asymptotic freedom is important to the understanding of the approximate scaling behavior in the deep inelastic lepton-nucleon scattering.

To compute $\beta(g)$ away from small g, we need other methods of calculation. Using the strong coupling lattice calculation at large g and the perturbation expansion at small g, Kogut and others obtained a possible sketch of $\beta(g)$ for a pure SU(3) gauge theory as shown in Fig. 8.6 (see Kogut, J. *Phys. Rep.* **67** (1980) 388).

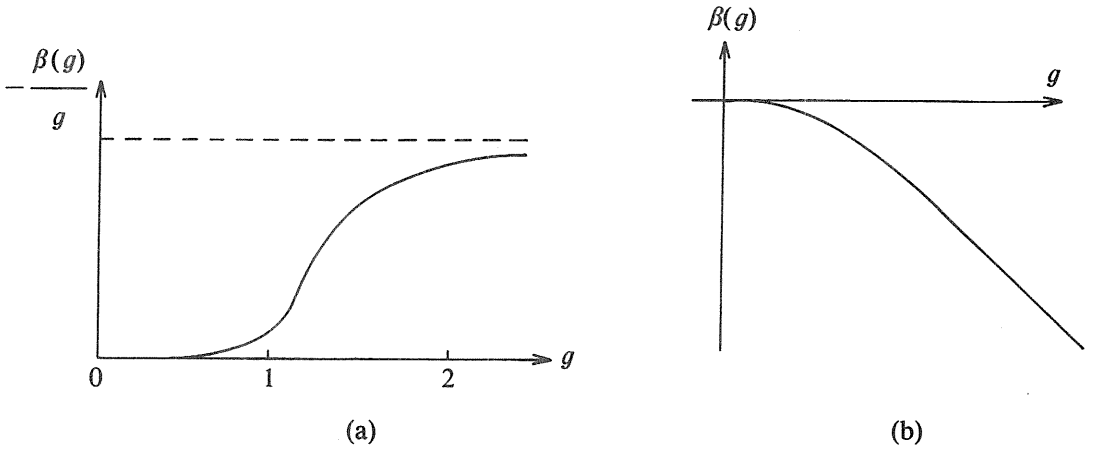


Fig. 8.6 Toy β-function for SU(3) gauge theory.

In the hydrodynamics analogy, β has the interpretation of the fluid velocity. Thus, as $p \to \infty$ and $t \to \infty$, the fluid near $g=0$ always converges to $\bar{g}=0$ as mentioned earlier. At the large distance, we consider the opposite limit $t \to -\infty$. If $\beta(g)$ indeed behaves as described in Fig. 8.6, the fluid will run away to $\bar{g}=\infty$ which leads to an infinitely strong running coupling constant. This ever increasing running coupling constant at large distance may be essential for the "color confinement" which prevents the quarks from becoming free particles.

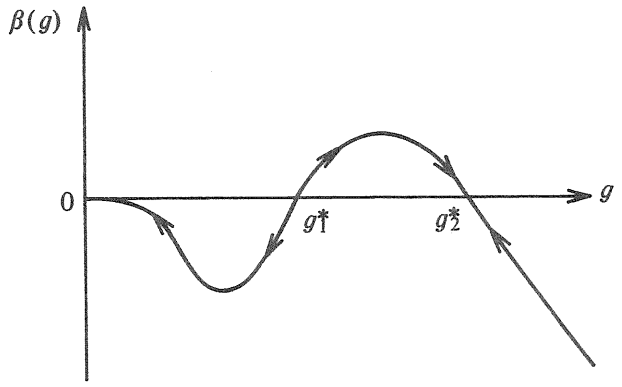


Fig. 8.7 A $\beta(g)$ function which has several fixed points.

The asymptotic behavior of a theory would be quite different if $\beta(g)$ behaves as in Fig. 8.7. Depending on whether the initial values of g is smaller or larger than g_1^*, the large p (ultraviolet) behavior of the running coupling constant $\bar{g}$ will approach either zero or g_2^*. In other words, we can lose our asymptotic freedom if the initial g is too large.

We can generalize our theory to include more than one coupling constant. The asymptotic behavior of the running coupling constants are determined by a multidimensional flow pattern. Knowing $\beta_i(\{g\})$ for all $\{g\}$, we can determine the fixed points and their basins of attraction. From the fixed points, we can obtain the asymptotic behavior of the Green's functions.

Chapter 9

INSTANTON

As we shall see, instanton physics is a semi-classical phenomenon. The Euclidean solutions that we shall describe are always classical solutions. actions. 't Hooft pointed out that these Euclidean solutions may be interpreted as the realization of tunnellings among inequivalent vacuums.

1. Solutions to a Classical Yang-Mills Theory

As we shall see, instanton physics is semi-classical phenomenon. The Euclidean solutions that we shall describe are always classical solutions.

The Yang-Mills Lagrange function in the matrix form is

$$\mathscr{L} = -\frac{1}{2}\operatorname{tr}(G_{\mu\nu}^2) \tag{9.1}$$

with

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{g}{i}[B_\mu, B_\nu]\,. \tag{9.2}$$

We shall rescale $G_{\mu\nu}$ and B_μ through

$$G'_{\mu\nu} = g G_{\mu\nu} \tag{9.3}$$

$$B'_\mu = g B_\mu. \tag{9.4}$$

Then we have

$$\mathscr{L} = -\frac{1}{2g^2}\operatorname{tr}(G'^2_{\mu\nu}) \tag{9.5}$$

and

$$G'_{\mu\nu} = \partial_\mu B'_\nu - \partial_\nu B'_\mu + \frac{1}{i}[B'_\mu, B'_\nu]\,. \tag{9.6}$$

In the following, we shall use the expressions (8.5) and (8.6), and omit the primes for simiplicity.

The Euclidean Hamiltonian $\mathscr{H}$ and the field tensor $G_{\mu\nu}$ obey the same equations as (8.5) and (8.6). However, the metric tensor $g_{\mu\nu}$ becomes

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{9.7}$$

The Euclidean Hamiltonian density $\mathscr{H}$ is always positive.

For Euclidean solutions, we can decompose $G_{\mu\nu}$ as the sum of a self-dual and an anti-selfdual part

$$G_{\mu\nu} = G^{(S)}_{\mu\nu} + G^{(A)}_{\mu\nu}, \tag{9.8}$$

where the self-dual part obeys

$$G^{(S)}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} G^{(S)}_{\lambda\rho} \tag{9.9}$$

and the anti-selfdual part obeys

$$G^{(A)}_{\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} G^{(A)}_{\lambda\rho}. \tag{9.10}$$

The Euclidean action is

$$\begin{aligned} S &= \int d^4x \mathscr{H}(x) \\ &= \frac{1}{2g^2}\int d^4x \left[\mathrm{tr}\,(G^{(S)}_{\mu\nu})^2 + \mathrm{tr}\,(G^{(A)}_{\mu\nu})^2\right] \qquad (9.11) \\ &\geq 0. \end{aligned}$$

If $G_{\mu\nu} \to 0$ as $|x| \to \infty$ so that B_μ is a pure gauge, then

$$\frac{1}{16\pi^2}\int d^4x\, \mathrm{tr}\,(G_{\mu\nu}\tilde{G}_{\mu\nu}) = n \tag{9.12}$$

is an integer, known as the winding number of the configuration. One can show that n is the topological winding number associated with the mapping of the (compactified) position space to the group space. In terms of $G^{(S)}$ and $G^{(A)}$, we can wirte (9.12) as

$$\frac{1}{16\pi^2}\int d^4x\,\mathrm{tr}\left[(G_{\mu\nu}^{(S)})^2-(G_{\mu\nu}^{(A)})^2\right]=n. \tag{9.13}$$

For a fixed winding number n, we can express the action S as

$$S=\frac{1}{g^2}\int d^4x\,\mathrm{tr}\,(G_{\mu\nu}^{(A)})^2+\frac{1}{2g^2}\int d^4x\left[\mathrm{tr}\,(G_{\mu\nu}^{(S)})^2-\mathrm{tr}\,(G_{\mu\nu}^{(A)})^2\right]$$

$$=\frac{1}{g^2}\int d^4x\,\mathrm{tr}\left[(G_{\mu\nu}^{(A)})^2\right]+\frac{8\pi^2 n}{g^2} \tag{9.14a}$$

$$=\frac{1}{g^2}\int d^4x\,\mathrm{tr}\,(G_{\mu\nu}^{(S)})^2+\frac{8\pi^2(-n)}{g^2}. \tag{9.14b}$$

Thus, S is bounded from below by $8\pi^2|n|/g^2$. For $n>0$, the self-dual solutions give the minimum S

$$S=\frac{8\pi^2 n}{g^2},\qquad n>0\qquad\text{(selfdual solution)} \tag{9.15a}$$

and for $n<0$, the anti-selfdual solutions give the minimum S

$$S=-\frac{8\pi^2 n}{g^2},\qquad n<0\qquad\text{(anit-selfdual solution).} \tag{9.15b}$$

Since S is a minimum for a self-dual or anti-selfdual solution, we expect

$$\delta S=0. \tag{9.16}$$

Equation (9.16) implies that B_μ and $G_{\mu\nu}$ obey the Euclidean equations

$$G_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu+\frac{1}{i}[B_\mu,B_\nu], \tag{9.17}$$

$$[\partial_\mu - iB_\mu, G_{\mu\nu}] = 0. \tag{9.18}$$

In other words, the self-dual and the anti-selfdual condition,

$$G_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} G_{\lambda\rho} \tag{9.19}$$

implies the full Yang-Mills equations (9.17), (9.18). Since the self-dual and the anti-selfual equations are linear in derivatives, they are easier to solve.

The $n=1$ self-dual solution obtained by Belavin *et al.* are

$$B_0 = \frac{\mathbf{x}\cdot\boldsymbol{\tau}}{\mathbf{x}^2+x_0^2+a^2} \tag{9.20}$$

$$\mathbf{B} = \frac{\mathbf{x}\times\boldsymbol{\tau} + x_0\boldsymbol{\tau}}{\mathbf{x}^2 + x_0^2 + a^2} \tag{9.21}$$

where $(x_0, \mathbf{x})$ are the Euclidean coordinates, $\boldsymbol{\tau}$ the Pauli matrix, and a a parameter describing the size of the instanton. It is straightforward to verify that the electric and the magnetic fields are

$$E_k \equiv G_{0k} = -\frac{2a^2}{(\mathbf{x}^2+x_0^2+a^2)^2}\tau_k \tag{9.22}$$

$$B_k \equiv \frac{1}{2}\varepsilon_{klm} G_{lm} = -\frac{2a^2}{(\mathbf{x}^2+x_0^2+a^2)^2}\tau_k \tag{9.23}$$

which indeed satisfy the self-dual condition.

The action S for this solution is

$$\begin{aligned} S &= \frac{1}{2g^2}\int d^4x\,[\mathrm{tr}\,(G_{\mu\nu}^2)] \\ &= \frac{1}{g^2}\int d^4x\,\mathrm{tr}\left[G_{0k}^2 + \frac{1}{2}G_{lm}^2\right] \\ &= \frac{48}{g^2}\int d^4x\,\frac{a^4}{(\mathbf{x}^2 + x_0^2 + a^2)^4} = \frac{8\pi^2}{g^2}. \end{aligned} \tag{9.24}$$

One can verify easily that the winding number is indeed one.

't Hooft interpreted the e^{-S} as the tunnelling amplitude among inequivalent vacuums. We shall use a simple quantum system to illustrate the connection between an instanton solution and its associated quantum tunneling.

2. Instanton as a Tunnelling Solution

We consider the WKB method in a two-dimensional quantum system described by

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - V(x, y) \tag{9.25}$$

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + V(x, y). \tag{9.26}$$

The time-independent Schroedinger euqation gives

$$\left[-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{2x^2} + \frac{\partial^2}{2y^2}\right) + V(x, y)\right]\psi(x, y) = E\psi(x, y). \tag{9.27}$$

We have kept $\hbar$ explicitly in (9.27). We apply WKB method to (9.27) by expanding ln $\psi(x, y)$ as a power series in $\hbar$, giving

$$\psi = Ae^{iS/\hbar}, \qquad S = S_0 + \hbar S_1 + \ldots. \tag{9.28}$$

Substituting (9.28) into (9.27) and equating the coefficients of $\hbar$, we obtain a set of equations

$$\frac{1}{2m}(\nabla S_0)^2 + V(x, y) = E, \tag{9.29}$$

$$-i\nabla^2 S_0 + 2\nabla S_0 \cdot \nabla S_1 = \mathrm{O}; \quad \text{etc.} \tag{9.30}$$

We shall concentrate on the lowest order equation (9.29). Just as in the optics, for every solution to (9.29), we can construct a ray representation as in Fig. 9.1. A ray is constructed as a path perpendicular to the family of curves $S_0(x, y) =$ constant. Along a ray, we have

$$(\nabla S_0)^2 = \left(\frac{dS_0}{ds}\right)^2 \tag{9.31}$$

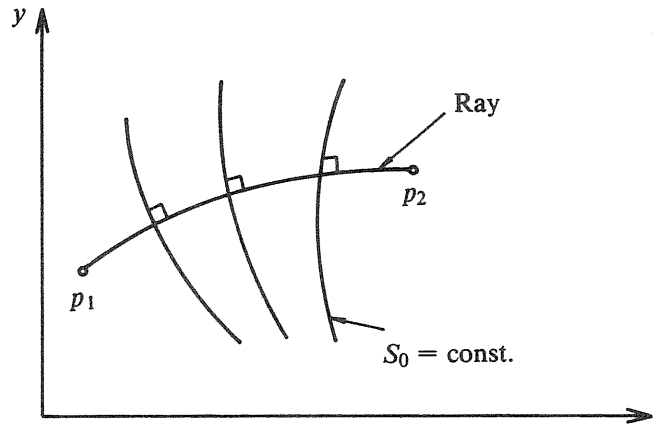


Fig. 9.1 The ray representation of $S_0(x, y)$.

where s is the length measured along the curve. Equation (9.29) then reduces to

$$\frac{dS_0}{ds} = \sqrt{2m(E - V)}$$

or

$$S_0 = \int_{P_1}^{P_2} ds \sqrt{2m(E - V(x, y))}. \tag{9.32}$$

The phase S_0 is stationary,

$$\delta S_0 = \delta \int_{P_1}^{P_2} ds \sqrt{2m(E - V(x-y))}$$

$$= \delta \int_{\lambda_1}^{\lambda_2} d\lambda \frac{ds}{d\lambda} \sqrt{2m(E - V(x, y))} = 0 \tag{9.33}$$

where λ is some parametrization. The variational principle implies

$$\frac{d}{d\lambda}\left\{\frac{\partial}{\partial\left(\frac{\partial x}{\partial\lambda}\right)}\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}\sqrt{2m(E-V)}\right\}$$

$$= \frac{\partial}{\partial x}\left[\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}\sqrt{2m(E-V)}\right] \tag{9.34a}$$

$$\frac{d}{d\lambda}\left\{\frac{\partial}{\partial\left(\frac{\partial y}{\partial x}\right)}\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}\sqrt{2m(E-V)}\right\}$$

$$=\frac{\partial}{\partial y}\left[\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}\sqrt{2m(E-V)}\right] \tag{9.34b}$$

and

$$\frac{ds}{d\lambda}=\sqrt{\left(\frac{\partial x}{\partial\lambda}\right)^2+\left(\frac{\partial y}{\partial\lambda}\right)^2}. \tag{9.35}$$

Equation (9.34a) gives

$$\frac{d}{d\lambda}\left[\sqrt{2m(E-V)}\,\frac{dx}{d\lambda}\Big/\frac{ds}{d\lambda}\right]=\frac{ds}{d\lambda}\frac{\left(-m\frac{\partial V}{\partial x}\right)}{\sqrt{2m(E-V)}}. \tag{9.36}$$

Introducing the time variable $t=t(\lambda)$ via

$$\frac{ds}{dt}=\frac{ds}{d\lambda}\frac{d\lambda}{dt}=\sqrt{\frac{2(E-V(x,y))}{m}}, \tag{9.37}$$

we have

$$m\frac{d}{d\lambda}\left(\frac{dx}{dt}\right)=-\left(\frac{d\lambda}{dt}\right)^{-1}\frac{\partial V}{\partial x}$$

or

$$m\frac{d^2x}{dt^2}=-\frac{\partial V}{\partial x}. \tag{9.38a}$$

Similarly, we have

$$m\frac{d^2y}{dt^2}=-\frac{\partial V}{\partial y}. \tag{9.38b}$$

In the tunnelling region, the action is imaginary. We can write ψ as

$$\psi = A\, e^{-R/\hbar},\ R = R_0 + \hbar R_1 + \cdots . \tag{9.39}$$

Equations (9.24) and (9.30) become

$$-\frac{1}{2m}(\nabla R_0)^2 + V(x, y) = E, \tag{9.40}$$

$$-\nabla^2 R + 2\nabla R_0 \cdot \nabla R_1 = 0, \text{ etc.} \tag{9.41}$$

In analogy to the rays in Fig. 9.1, we can construct paths which are perpendicular to the family of curves R = constant. Along such paths, R is stationary, and is often a minimum. If the system tunnels from one point to another along a minimum R curve, the tunnelling rate is maximum. Banks, Bender and Wu (*Phys. Rev.* **D8**, (1973) 3346) referred to these paths as the most probable escape (tunnelling) paths (MPEP). Along an MPEP, we obtain from (9.40)

$$R_0 = \int_{P_1}^{P_2} ds \sqrt{2m(V - E)} \tag{9.42}$$

and

$$\delta R_0 = 0. \tag{9.43}$$

Equations (9.42) and (9.43) are the same as (9.33) with $V \to -V$ and $E \to -E$. Hence, with a proper parametrization $x = x(\tau)$, $y = y(\tau)$, we find that the MPEP obeys

$$m\frac{d^2x}{d\tau^2} = \frac{\partial V}{\partial x},\ m\frac{d^2y}{d\tau^2} = \frac{\partial V}{\partial y}. \tag{9.44}$$

Equation (9.44) is the same as (9.38) with $t \to i\tau$. For this reason, the tunnelling parameter τ is also called the Euclidean or imaginary time. In spite of its "imaginary" label, the variable τ is always real and describes the intermediate configurations of the tunnelling particle along an MPEP. Once $x(\tau)$ and $y(\tau)$ are determined, we can reduce our problem to an effective one-dimensional quantum mechanics problem which one can solve straightforwardly.

To construct the tunnelling path, we can also start directly from the variational principle with the Lagrangian in Eq. (9.25),

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y). \tag{9.25}$$

We represent the unknown tunnelling path as $(x(\lambda), y(\lambda))$ where λ is an arbitrary parametrization. Along the MPEP, we can express L in (9.25) by an effective one-dimensional Lagrangian

$$L_{\text{eff}} = \frac{m}{2}\left[\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2\right]\dot{\lambda}^2 - V(x, y)$$

$$= \frac{1}{2} m(\lambda)\,\dot{\lambda}^2 - V(\lambda) \tag{9.45}$$

where

$$m(\lambda) \equiv \frac{m}{2}\left[\left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2\right] \tag{9.46}$$

$$V(\lambda) \equiv V(x(\lambda), y(\lambda)). \tag{9.47}$$

To the leading WKB approximation, we do not need to worry about the ordering of $m(\lambda)$ and $\dot{\lambda}^2$ in (9.43). The effective one-dimensional Hamiltonian, within the same ordering ambiguity, is

$$H_{\text{eff}} = P_\lambda^2/2m(\lambda) + V(\lambda) \tag{9.48}$$

with

$$P_\lambda = m(\lambda)\,\dot{\lambda}\ . \tag{9.49}$$

The WKB tunnelling amplitude based on this effective one-dimensional theory is the same $e^{-R_0/\hbar}$ with

$$R_0 = \int d\lambda \sqrt{2m(V(\lambda) - E)}\,. \tag{9.50}$$

Note that R_0 in (9.50) is independent of the parametrization λ. The variational principle $\delta R_0 = 0$ gives rise to the desired Euclidean equations and the MPEP.

3. Tunnelling Potential and θ Vacuum

Now we return to study the Yang-Mills instanton as a most probable escape path (MPEP). To obtain the MPEP from an instanton, we only need to replace the Euclidean time x_0 by a parameter $\lambda(t)$ where t is the Minkowski time. This indeed gives the correct answer in the temporal gauge ($A_0=0$). In an arbitrary gauge, we only need to make a slight modification.

Consider the Yang-Mills Lagrange function in the 3+1 Minkowski space.

$$\mathscr{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,(G_{\mu\nu})^2 \tag{9.51}$$

and

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + \frac{1}{i}[B_\mu, B_\nu]\,. \tag{9.52}$$

We write down the MPEP as

$$B_0 = \frac{\mathbf{x}\cdot\boldsymbol{\tau}}{\mathbf{x}^2+\lambda^2+a^2}\,\dot{\lambda}(t) \tag{9.53}$$

$$\mathbf{B} = \frac{\lambda\boldsymbol{\tau} + \mathbf{x}\times\boldsymbol{\tau}}{\mathbf{x}^2+\lambda^2+a^2}\,. \tag{9.54}$$

From (9.53) and (9.54), we can compute the field tensor as

$$E_k \equiv G_{0k} = -\frac{2a^2\dot{\lambda}}{(\mathbf{x}^2+\lambda^2+a^2)^2}\,\tau_k\,. \tag{9.55}$$

$$B_k \equiv \frac{1}{2}\,\varepsilon_{klm}\,G_{lm} = -\frac{2a^2}{(\mathbf{x}^2+\lambda^2+a^2)^2}\,\tau_k\,. \tag{9.56}$$

Along an MPEP, the electric and the magnetic fields are parallel to each other, in both the position and the group space. We can introduce the winding number as

$$\Delta Q = \frac{1}{32\pi^2}\,\mathrm{tr}\int_{\sigma_1}^{\sigma_2} d^4x\,\varepsilon_{\mu\nu\lambda\sigma}\,G_{\mu\nu}\,G_{\lambda\sigma}$$

$$= \frac{1}{4\pi^2}\,\mathrm{tr}\int d^4x\,\mathbf{E}\cdot\mathbf{B}$$

$$= \frac{1}{4\pi^2}\,\mathrm{tr}\int d^4x\left[\frac{3\,(2a^2)^2}{(\mathbf{x}^2+\lambda^2+a^2)^4}\,\dot{\lambda}\right]$$

$$= \frac{24}{4\pi^2}\int d^3x\,d\lambda\,\frac{a^4}{(\mathbf{x}^2+\lambda^2+a^2)^4}$$

$$= 1. \tag{9.57}$$

It is straightforward to show that ΔQ is the difference of two surface integrals, independent of the intermediate dynamics. This follows from the relation

$$\frac{1}{32\pi^2}\,\varepsilon_{\mu\nu\lambda\sigma}\,\mathrm{tr}\,(G_{\mu\nu}\,G_{\lambda\sigma}) = \partial_\mu K_\mu \tag{9.58}$$

with

$$K_\mu = \frac{\varepsilon_{\mu\nu\lambda\sigma}}{8\pi^2}\,\mathrm{tr}\left(B_\nu\partial_\lambda B_\sigma - \frac{2i}{3}\,B_\nu B_\lambda B_\sigma\right). \tag{9.59}$$

We can associate the surface (d^3x) integration of K_0 as a topological charge. Then, ΔQ is the difference of two topological integrals,

$$\Delta Q = \int_{t_2} - \int_{t_1} d^3x\,K_0 = Q(t_2) - Q(t_1). \tag{9.60}$$

Thus, along this MPEP, the net change of winding number is 1. As we have mentioned earlier, the MPEP defines an effective one-dimensional tunnelling path. Along such a path, the system is one-dimensional. We can substitute (9.53) and (9.54) in $\mathscr{L}$ to obtain an effective Lagrangian

$$
\begin{aligned}
L_{\text{eff}} &= \int d^3x \, \mathscr{L}(B_\mu(\lambda(t), \mathbf{x})) \\
&= \frac{1}{g^2} \int d^3x \, \text{tr} \, (\mathbf{E}^2 - \mathbf{B}^2) \\
&= \frac{6}{g^2} \int d^3x \frac{(2a^2)^2}{(\mathbf{x}^2 + \lambda^2 + a^2)^4} (\dot{\lambda}^2 - 1) \\
&= \frac{3\pi^2 a^4}{g^2 (\lambda^2 + a^2)^{5/2}} \dot{\lambda}^2 - \frac{3\pi^2 a^4}{g^2 (\lambda^2 + a^2)^{5/2}} \\
&= \frac{1}{2} m(\lambda) \dot{\lambda}^2 - V(\lambda) \qquad (9.61)
\end{aligned}
$$

with

$$
m(\lambda) = \frac{6\pi^2 a^4}{g^2 (\lambda^2 + a^2)^{5/2}} \qquad (9.62)
$$

$$
V(\lambda) = \frac{3\pi^2 a^4}{g^2 (\lambda^2 + a^2)^{5/2}} . \qquad (9.63)
$$

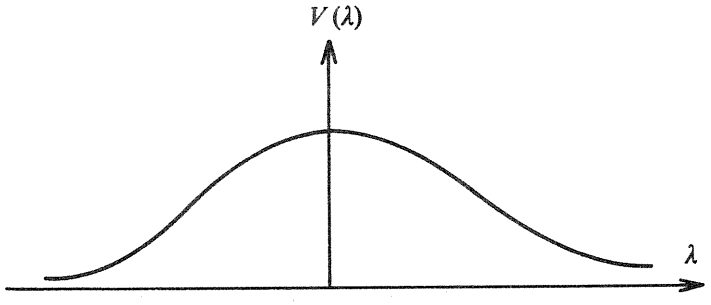


Fig. 9.2 The effective potential barrier in λ-space.

Both **E** and **B** fields vanish as $\lambda \to \pm\infty$, which correspond to two inequivalent classical ground states. In between, the potential energy $V(\lambda)$ is positive and serves as a barrier (Fig. 9.2). We can compute the WKB tunnelling amplitude as before. The WKB tunnelling exponent between the vacuums (classical energy = 0) is

$$R = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{2m(\lambda)\,(V(\lambda) - 0)}$$

$$= \int_{-\infty}^{\infty} d\lambda \frac{6\pi^2 a^4}{g^2\,(\lambda^2+a^2)^{5/2}}$$

$$= \frac{8\pi^2}{g^2}. \tag{9.64}$$

Thus, the tunnelling amplitude is indeed

$$e^{-R} = \exp\left(\frac{-8\pi^2}{g^2}\right). \tag{9.65}$$

as interpreted by 't Hooft.

We can establish the following conclusion based on semiclassical treatment in the Minkowski space: the WKB vacuum tunnelling exponent R is bounded below by $8\pi^2\,|n|/g^2$ where n is the net change of winding number on the tunnelling process. The lower bound is achieved if, and only if, the electric and the magnetic fields are proportional to each other in both the position and the group space,

$$\mathbf{E}(x, t) = \mathbf{B}(x, t)\, f(t) \tag{9.66}$$

and if the proportionality constant $f(t)$ is x-independent. Even though $\mathbf{E}$ and $\mathbf{B}$ are not gauge invariant, the condition $\mathbf{E}//\mathbf{B}$ is. One can verify easily that the condition $\mathbf{E}//\mathbf{B}$ is also independent of the reparametrization of λ.

We can used the reparametrization property to change the mass parameter $m(\lambda)$. Consider a new parametrization,

$$q = q(\lambda),\ \lambda = \lambda(q), \tag{9.67}$$

we have

$$L_{\text{eff}} = \frac{1}{2} m(\lambda) \left(\frac{d\lambda}{dq}\right)^2 \dot{q}^2 - V(\lambda(q))$$

$$= \frac{1}{2} m(q)\, \dot{q}^2 - V(q) \tag{9.68}$$

where

$$V(q) = V(\lambda(q)) \tag{9.69}$$

$$m(q) = m(\lambda)\left(\frac{d\lambda}{dq}\right)^2. \tag{9.70}$$

The new momentum is

$$p_q = m(q)\,\dot{q} = \frac{d\lambda}{dq}\,p_\lambda. \tag{9.71}$$

The WKB exponent

$$\int p_q\,dq = \int p_\lambda d\lambda \tag{9.72}$$

is not affected by the reparametrization. We now introduce a winding number variable q such that

$$q(-\infty) = 0, \quad q(\infty) = 1 \tag{9.73}$$

and

$$m(q) = m, \text{ a constant.} \tag{9.74}$$

Equations (9.73) and (9.74) determine $q(\lambda)$ uniquely as

$$\begin{aligned} q(\lambda) &= \int_{-\infty}^{\lambda} d\lambda \sqrt{\frac{m(\lambda)}{m}} \\ &= \int_{-\infty}^{\lambda} d\lambda \sqrt{\frac{6\pi^2 a^4}{mg^2\,(\lambda^2+a^2)^{5/2}}} \\ &= \frac{\Gamma(5/4)}{\Gamma(1/2)\,\Gamma(3/4)} \int_{-\infty}^{\lambda/a} \frac{dt}{(t^2+1)^{5/4}} \end{aligned} \tag{9.75}$$

and

$$m \equiv \frac{6\pi^2 a}{g^2}\left(\frac{\Gamma(1/2)\,\Gamma(3/4)}{\Gamma(5/4)}\right)^2. \tag{9.76}$$

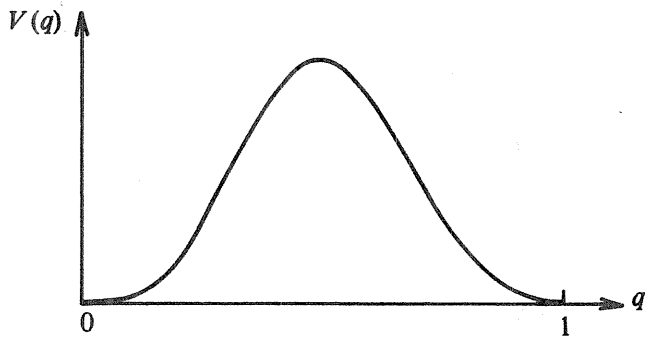


Fig. 9.3 The effective potential barrier in winding number space.

In term the winding number q, the potential barrier is a smooth potential as shown in Fig. 9.3.

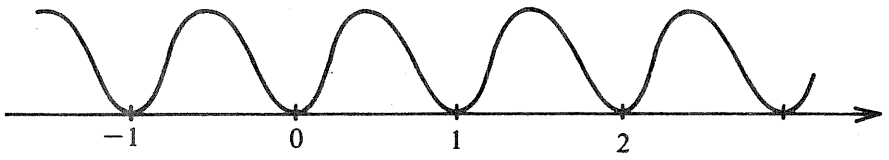


Fig. 9.4 The periodic vacuum energy barriers.

Of course, the winding number can change more than one unit. The tunnelling potential will be a periodic function in q, as shown in Fig. 9.4. This is analogous to a particle moving in a periodic potential — a problem well-studied in solid state physics. The energy eigenstate for such a periodic potential is a Bloch-wave, obeying

$$\psi(q + 1) = e^{i\theta}\psi(q). \tag{9.77}$$

This energy eigenstate with a given θ is called a θ-vacuum. Vacuum states with different θ's are orthogonal to each other.

4. Zero Modes

Consider a set of massless fermions coupled to a Yang-Mills field in the Euclidean space,

$$\mathscr{L} = -\frac{1}{4} G_{\mu\nu} \cdot G_{\mu\nu} - \bar{\psi}\gamma_\mu D_\mu \psi. \tag{9.78}$$

If $\mathbf{B}_\mu$ describes an instanton solution, then, ψ contains necessarily a zero mode solution,

$$\gamma \cdot D\psi = 0. \tag{9.79}$$

The following proof was given by Brown *et al.* They first introduce

$$T(m^2) \equiv \mathrm{Tr}\left(\frac{m}{\not{D}+m}\gamma_5\right) = \mathrm{Tr}\left(\frac{m^2}{-\not{D}^2+m^2}\gamma_5\right). \tag{9.80}$$

where Tr denotes trace over the Euclidean space (x), the spin space, and the internal space (isospin). In (9.80), m^2 is an arbitrary mass parameter. One can show that $T(m^2)$ is independent of m^2:

$$\begin{aligned}
\frac{\partial}{\partial m^2} T(m^2) &= \mathrm{Tr}\left[\frac{-\not{D}^2}{(-\not{D}^2+m^2)^2}\gamma_5\right] \\
&= \mathrm{Tr}\left[\frac{\not{D}}{(-\not{D}^2+m^2)^2}\gamma_5\not{D}\right] && \text{(move } \not{D} \text{ through } \gamma_5\text{)} \\
&= \mathrm{Tr}\left[\frac{\not{D}^2}{(-\not{D}^2+m^2)^2}\gamma_5\right] && \text{(cyclic property of Tr)} \\
&= 0. && (9.81)
\end{aligned}$$

Next, we evaluate $T(m^2)$ at $m \to \infty$. Using

$$\begin{aligned}
\not{D}^2 &= D^2 + \frac{1}{4}[\gamma_\mu, \gamma_\nu][D_\mu, D_\nu] \\
&= D^2 + \frac{1}{2}\tau_{\mu\nu}\left(G^a_{\mu\nu}\cdot\frac{\tau^a}{2}\right) && (9.82) \\
&= D^2 + \frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu}\,,
\end{aligned}$$

we have

$$\lim_{m\to\infty} T(m^2) = \lim_{m\to\infty} \mathrm{Tr}\left[\frac{m^2}{-D^2+m^2-\frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu}}\gamma_5\right]$$

$$= \lim_{m\to\infty} \mathrm{Tr}\left[\frac{m^2}{-D^2+m^2}\gamma_5 + \frac{m^2}{-D^2+m^2}\frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu}\frac{1}{-D^2+m^2}\gamma_5\right.$$

$$+\frac{m^2}{-D^2+m^2}\frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu}\frac{1}{-D^2+m^2}\frac{1}{2}\sigma_{\lambda\rho}G_{\lambda\rho}\cdot\frac{1}{-D^2+m^2}\gamma_5\Bigg]+O\left(\frac{1}{m^2}\right) \tag{9.83}$$

where the ignored terms contain $1/m^6$ and the momentum integration can provide at most m^4. The first two terms in (9.83) vanish identically due to the trace properties of γ_5. The third term survives, giving

$$\begin{aligned}
\lim_{m\to\infty} T(m^2) &= \mathrm{Tr}\left[\frac{1}{2}\sigma_{\mu\nu}G_{\mu\nu}\frac{1}{-D^2+m^2}\frac{1}{2}\sigma_{\lambda\rho}G_{\lambda\rho}\gamma_5\frac{m^2}{(-D^2+m^2)^2}\right] \\
&= m^2\,\varepsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}\int d^4x\,d^4y\,G_{\mu\nu}(x)\,D(x-y) \\
&\quad\times G_{\lambda\rho}(y)\,D^2(y-x) \\
&\approx m^2\,\varepsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}\int d^4x\,G_{\mu\nu}(x)\,G_{\lambda\rho}(x) \\
&\quad\times\int d^4y\,D(x-y)\,D^2(y-x) \\
&= \varepsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}\int d^4x\,G_{\mu\nu}(x)\,G_{\lambda\rho}(x) \\
&\quad\times\int\frac{d^4p}{(2\pi)^4}\frac{m^2}{(p^2+m^2)^3} \\
&= \frac{1}{32\pi^2}\int d^4x\,\mathrm{tr}\,(\varepsilon_{\mu\nu\lambda\rho}\,G_{\mu\nu}\,G_{\mu\rho}) \\
&= \text{winding number } Q
\end{aligned} \tag{9.84}$$

where we have replaced the Green's function $(-D^2+m^2)^{-1}$ by $(-\partial^2+m^2)^{-1}$ in (9.84). On the other hand, except for the zero modes, $T(m^2)$ vanishes as $m\to 0$, giving

$$\lim_{m\to 0} T(m^2) = n_+ - n_- \tag{9.85}$$

where n_+ and n_- are the numbers of $\gamma_5 = \pm 1$ zero modes of $\not{D}$. Hence, we have

$$n_+ - n_- = Q, \text{ winding number.} \tag{9.86}$$

Thus, if $Q \neq 0$, zero modes must exist. The above derivation is known in mathematics as the index theorem.

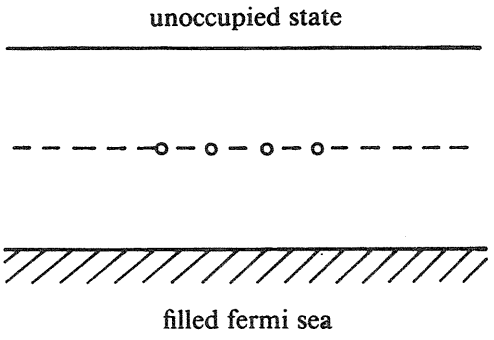


Fig. 9.5 Zero modes and degenerate vacuum states.

In the Minkowski space, if we look at the fermions in the ground state, we shall see that at $t \to \pm\infty$, these zero modes lead to zero energy fermion states sitting in the gap between the unoccupied and the occupied states. We denote the occupation numbers for such states by $n_\pm$where $+/-$ describe the chirality of these states. These zero energy fermion states provide some degeneracies of the ground states. (See Fig. 9.5.) As the vacuum tunnels from $Q = 0$ to $Q = 1$ ($\Delta Q = 1$), we find that n_+ changes from 0 to 1 or n_- changes from 1 to 0. In the former case, an unoccupied $\gamma_5 = 1$ zero energy state is now occupied. We may interpret this process as the emission of a $\gamma_5 = 1$ zero energy fermion. In the latter case, we encounter the absorption of a $\gamma_5 = -1$ zero energy antifermion. This emission and absorption should occur for each of the massless fermions which are coupled to the gauge field.

To see the quantitative effect of the zero modes, we consider the vacuum to vacuum transition amplitude in the presence of an external source $\bar{\psi}\vartheta\psi$,

$$\begin{aligned} \langle 0|0\rangle^{\vartheta} &= \int \mathscr{D}\{B\}\,\mathscr{D}\{\psi\}\,\mathscr{D}\{\bar{\psi}\}\, e^{\int d^4x\,(\mathscr{L}(B) + \bar{\psi}\vartheta\psi)} \\ &= \int \mathscr{D}\{B\}\, e^{\int d^4x\,\mathscr{L}\{B\}} \det \mathscr{M} \end{aligned} \tag{9.87}$$

with

$$\mathcal{M} \equiv \not{D} + \vartheta. \tag{9.88}$$

In the absence of ϑ, we have

$$\det \mathcal{M} = \lambda_1 \times \cdots \times \lambda_N \det' \mathcal{M} = 0 \tag{9.89}$$

where $\lambda_i = 0$ are the eigenvalues associated with the zero modes and $\det' \mathcal{M}$ does not vanish at $\vartheta = 0$. The vanishing of λ_i's implies

$$\langle 0 | 0 \rangle^{\vartheta=0} = 0. \tag{9.90}$$

Note that $\langle 0 | 0 \rangle = 0$ does not indicate that the vacuum tunnelling is forbidden. It implies that zero-energy fermions $(q, \bar{q})$ are present in the final state

$$\langle q, \bar{q}, \cdots | 0 \rangle \approx e^{-8\pi^2/g^2} \neq 0. \tag{9.91}$$

To compute (9.91), we evaluate (9.89) for small but nonvanishing ϑ,

$$\det \mathcal{M}(\vartheta) = \lambda_1(\vartheta) \times \cdots \times \lambda_N(\vartheta) \det' \mathcal{M} \tag{9.92}$$

where $\det' \mathcal{M}$ is essentially unchanged and $\lambda_i(\vartheta)$ are now nonzero. Then, we have

$$\langle 0 | 0 \rangle^{\vartheta} = \int \mathcal{D}[B]\, e^{\int d^4x \mathcal{L}(B)} \det' \mathcal{M} \times \lambda_1(\vartheta) \times \cdots \times \lambda_N(\vartheta)$$

$$= \text{const}\, e^{-8\pi^2 |n|/g^2} \prod \lambda_i(\vartheta). \tag{9.93}$$

For small ϑ, $\lambda_i(\vartheta)$ are linear in ϑ. Expanding (9.93) in ϑ, we obtain the detailed expression of (9.91). The number of λ factors in (9.93) is the same as the number of massless fermions (flavor number) coupled to the gauge field. For N flavors of massless fermions, the vacuum tunnelling induces the production and/or absorption of $2N$ of these zero-mode fermions. These zero-mode fermions are chiral eigenstates.
Let

$$j_\mu^5 = \sum_f \bar{\psi}_f \gamma_\mu \gamma_5 \psi_f \tag{9.94}$$

be the axial current associated with the massless fermions. We can establish that

$$\partial_\mu j_\mu^5 = \frac{N}{32\pi^2}\,\varepsilon_{\mu\nu\lambda\rho}\,\mathrm{tr}\,(G_{\mu\nu}G_{\lambda\rho}) \tag{9.95}$$

where N is the flavor number, Equation (9.95) is the famous axial anomaly. The chiral zero modes provide a physical realization of (9.95).

We shall review briefly 't Hooft's calculation of $\lambda(\vartheta)$ and its induced interaction. We shall omit the intermediate algebra, and present only the results.

We can express the one-instanton solution in (9.21) as

$$B_\mu^a(x) = \frac{2}{g}\,\frac{\eta_{\mu\nu}^a (x - x_0)_\nu}{(x - x_0)^2 + a^2}, \tag{9.96}$$

where $(x_0)_\mu$ is the origin of the instanton and

$$\left.\begin{aligned} \eta_{ij}^a &= \varepsilon_{aij} \\ \eta_{i0}^a &= \delta_{ai} \\ \eta_{00}^a &= 0 \end{aligned}\right\}. \tag{9.97}$$

Note that B_μ^a is the original Yang-Mills field before scaling described in Eqs. (9.3) and (9.4).

The zero-mode equation is

$$\gamma_\mu D_\mu \psi = 0. \tag{9.98}$$

The dependence on x_0 and a is trivial. We set them as $x_0 = 0$, $a = 1$. The solution to (9.98) is

$$\psi = (1 + r^2)^{-3/2}\, u(s)\, \sqrt{2}/\pi \tag{9.99}$$

where u obeys

$$\gamma_5 u = u \tag{9.100}$$

$$\left(\frac{\tau^a}{2} - \frac{i}{8}\,\eta_{\mu\nu}^a \gamma_\mu \gamma_\nu\right) u = 0. \tag{9.101}$$

This solution is localized in four-dimensional space, and has a given helicity ($\gamma_5 = 1$). Its spin and isospin are coupled in (9.101). The coupling between spin and isospin is a general feature of instanton and monopole solutions.

In the presence of the source term, we need to solve

$$(\gamma_\mu D_\mu + \vartheta(x))\,\psi(x) = \lambda(\vartheta)\,\psi. \tag{9.102}$$

If we only have one flavor, we have a single zero mode. The lowest order $\lambda(\vartheta)$ is

$$\lambda(\vartheta) = \int d^4x\, \psi^*(x)\, \vartheta\psi(x) \tag{9.103}$$

where $\psi(x)$ is the zero mode wave function. If we have N favors, we encounter N zero modes. The product of λ's is

$$\lambda_1(\theta)\,\lambda_2(\theta)\cdots\lambda_N(\theta) = \int d^4x \det(\psi^*\vartheta\psi) \tag{9.104}$$

with

$$\det(\psi\vartheta\psi) = \begin{vmatrix} \psi^*\vartheta_{11}\psi & \psi^*\vartheta_{12}\psi & \cdots \\ \psi\vartheta_{21}\psi & \psi^*\vartheta_{22}\psi & \cdots \\ \cdots & & \cdots \end{vmatrix} \tag{9.105}$$

where the label on ϑ are the flavor indices, and the ψ's are the zero mode wave functions.

Knowing λ's, we can construct an effective (zero mode) fermion interaction by requiring

$$\int \mathscr{D}\{\psi\}\,\mathscr{D}\{\bar\psi\} \int d^4y\, \mathscr{L}_{\text{eff}}(y)\, e^{\int d^4x\, \bar\psi\vartheta\psi} = \Pi\lambda(\vartheta). \tag{9.106}$$

We choose $\mathscr{L}_{\text{eff}}$ to be bilinear in $\bar\psi\psi$ for one flavor case, quadratic in $\bar\psi$ and ψ for two flavors, etc. For one-flavor case, we find

$$\mathscr{L}_{\text{eff}}(y) = C(y - x_0)\,\bar\psi(y)\frac{1+\gamma_5}{2}\psi(y) \tag{9.107}$$

with

$$C(r) = \frac{2}{\pi^2} \frac{1}{(1+r^2)^3}. \tag{9.108}$$

Multiplying (9.107) by $\exp(-8\pi^2/g^2)$, we have the effective Lagrange function for zero-mode fermion production due to an instanton at x_0. We can work out the $\mathscr{L}_{\text{eff}}$ for two and more flavor cases in a similar way.

The calculation done by 't Hooft indicates that proton will decay in many unification theories with instantons. Due to the exponential damping factor $\exp(-8\pi^2/g^2)$ and the small multiparticle phase space, the decay rate is too small to be of any practical interest.

5. Axial Anomaly

The expression $\frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$ appear naturally in the axial anomaly. We consider a QED type Lagrange function,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}{}^2 + \bar{\psi}[i\gamma_\mu(\partial_\mu + ieA_\mu) - m]\psi. \tag{9.109}$$

We introduce the vector, the axial-vector, and the pseudo-scalar currents as

$$j_\mu = \bar{\psi}\gamma_\mu\psi \tag{9.110}$$

$$j_{5\mu} = \bar{\psi}\gamma_\mu\gamma_5\psi \tag{9.111}$$

$$j_5 = \bar{\psi}i\gamma_5\psi. \tag{9.112}$$

The field equations are

$$\partial_\nu F_{\mu\nu} = e\bar{\psi}\gamma_\nu\psi \tag{9.113}$$

$$(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi = 0 \tag{9.114}$$

$$\bar{\psi}(-i\overleftarrow{\partial\!\!\!/} - eA\!\!\!/ - m) = 0. \tag{9.115}$$

From (9.114) and (9.115), we can show easily that

$$\partial_\mu j_\mu = 0. \tag{9.116}$$

If we apply (9.114) and (9.115) to the divergence of the axial current, we have formally

$$\begin{aligned}\partial_\mu j_{5\mu} &= \bar{\psi}\overleftarrow{\not\partial}\gamma_5\psi - \bar{\psi}\gamma_5\not\partial\psi \\ &= \bar{\psi}(e\not A + m)\, i\gamma_5\psi + \bar{\psi}i\gamma_5\,(e\not A + m)\,\psi \\ &= 2m\bar{\psi}i\gamma_5\psi \\ &= 2mj_5(x) \qquad \text{(false)}.\end{aligned} \tag{9.117}$$

As we shall see, Eq. (9.117) is false. There is an important term missing in (9.117).

The correct expression of (9.117) is

$$\partial_\mu j_{5\mu} = 2mj_5 + \frac{e^2}{16\pi^2}\,\varepsilon_{\mu\nu\lambda\rho}\,F_{\mu\nu}\,F_{\lambda\rho}. \tag{9.118}$$

We can extend this result to the electroweak and the strong interaction. According to the partially conserved axial-current hypothesis, the divergence of the axial current is related to the pion field. The additional term is responsible for $\pi^0 \to 2\gamma$ decay.

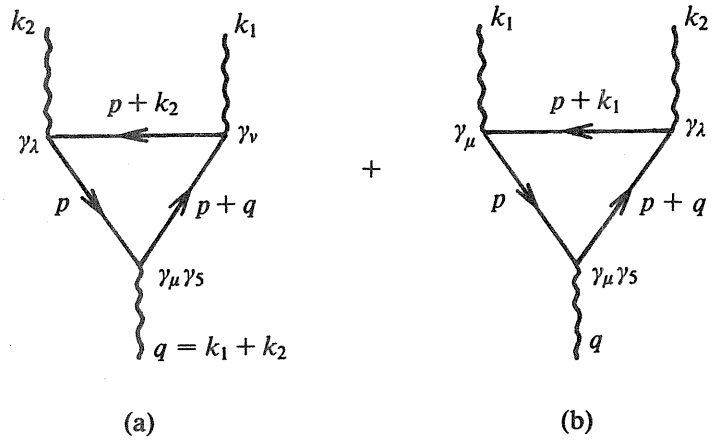


Fig. 9.6 Triangle diagrams associated with the axial anomaly.

To see the existence of the last term in (9.118), we evaluate the matrix element,

$$\langle\, 2\gamma \,|\, \partial_\mu j_{5\mu} \,|\, 0\,\rangle = \partial_\mu \langle\, 2\gamma | j_{5\mu} | 0\rangle. \tag{9.119}$$

The axial current matrix elements associated with Fig. 9.6a and b are

$$T^{(a)}_{\mu\nu\lambda}(k_1, k_2) = (-1)\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\gamma_\nu \frac{i}{\not p+\not q-m+i\varepsilon}\gamma_\mu\gamma_5 \times \frac{i}{\not p-m+i\varepsilon}\gamma_\lambda \frac{i}{\not p+\not k_2-m+i\varepsilon}\right], \tag{9.120}$$

$$T^{(b)}_{\mu\nu\lambda}(k_1, k_2) = T^{(a)}_{\mu\lambda\nu}(k_2, k_1)\,. \tag{9.121}$$

The matrix element of the divergence of the axial current associated with (9.120) is

$$iq_\mu\, T^{(a)}_{\mu\nu\lambda}(k_1, k_2) = (-1)\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\gamma_\mu \frac{i}{\not p+\not q-m+i\varepsilon} \times iq_\mu\gamma_\mu\gamma_5 \frac{i}{\not p-m+i\varepsilon}\gamma_\lambda \frac{i}{\not p+\not k_2-m+i\varepsilon}\right]. \tag{9.122}$$

We can express $iq_\mu\gamma_\mu\gamma_5$ as

$$iq_\mu\gamma_\mu\gamma_5 = i(\not p+\not q-m)\,\gamma_5 + i\gamma_5(\not p-m) + 2im\,\gamma_5\,. \tag{9.123}$$

The last term, $2im\gamma_5$, gives rise to the matrix element of $2mj_5(x)$. The first two terms give rise to the anomaly

$$\begin{aligned}(iq_\mu T_{\mu\nu\lambda})_{\text{anomaly}} &= \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\gamma_\nu\gamma_5 \frac{i}{\not p - m + i\varepsilon}\gamma_\lambda \frac{i}{\not p + \not k_2 - m + i\varepsilon}\right.\\ &\quad + \gamma_\nu \frac{i}{\not p + \not q - m + i\varepsilon}\gamma_5\,\gamma_\lambda \frac{i}{\not p + \not k_2 - m + i\varepsilon}\\ &\quad + \gamma_\nu\,\gamma_5 \frac{i}{\not p - m + i\varepsilon}\gamma_\nu \frac{i}{\not p + \not k_1 - m + i\varepsilon}\\ &\quad \left. + \gamma_\lambda \frac{i}{\not p + \not q - m + i\varepsilon}\gamma_5\,\gamma_\nu \frac{i}{\not p + \not k_1 - m + i\varepsilon}\right]\\ &= \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left[\gamma_\nu\gamma_5 \frac{i}{\not p - m + i\varepsilon}\gamma_\lambda \frac{i}{\not p + \not k_2 - m + i\varepsilon}\right.\end{aligned}$$

$$
\begin{aligned}
&- \gamma_\nu \gamma_5 \frac{i}{\not p + \not k_1 - m + i\varepsilon} \gamma_\lambda \frac{i}{\not p + \not k_1 + \not k_2 - m + i\varepsilon} \\
&+ \gamma_\lambda \gamma_5 \frac{i}{\not p - m + i\varepsilon} \gamma_\nu \frac{i}{\not p + \not k_1 - m + i\varepsilon} \\
&- \gamma_\lambda \gamma_5 \frac{i}{\not p + \not k_2 - m + i\varepsilon} \gamma_\nu \frac{i}{\not p + \not k_1 + \not k_2 - m + i\varepsilon} \Big] .
\end{aligned}
\tag{9.124}
$$

Equation (9.124) would vanish if we could make a shift of the integration variable. However, the individual integrals are quadratically divergent. The quadratical divergent parts cancel. The linearly divergent parts lead to a finite result

$$
(iq_\mu T_{\mu\nu\lambda})_{\text{anomaly}} = \frac{1}{4\pi^2} \varepsilon_{\mu\nu\lambda\rho} (k_1)_\rho (k_2)_\mu \tag{9.125}
$$

which implies

$$
(\partial_\mu j_{5\mu})_{\text{anomaly}} = \frac{e^2}{16\pi^2} \varepsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \tag{9.126}
$$

as desired.

It is interesting to know that higher order radiative corrections to the anomaly vanish. (See Adler and Bardeen.) This is because, in the higher order terms, the loop integrals become convergent. There are no contributions due to a shift of the integration variable.

In QCD, we have in the chiral limit

$$
\partial_\mu j_{5\mu} = \frac{Ng^2}{32\pi^2} \varepsilon_{\mu\nu\lambda\rho} \mathbf{G}_{\mu\nu} \cdot \mathbf{G}_{\lambda\rho} \tag{9.127}
$$

where N is the flavor number. Equation (9.127) implies that the divergence of the chiral current is related to the instanton density. In particular, the chiral charge Q_5 is not conserved, but is given by the instanton number n,

$$
\Delta Q_5 = 2Nn \tag{9.128}
$$

as described in the previous section.

6. Decay of A False Vacuum

We shall use the instanton technique to study the decay of a false vacuum. Consider the Lagrange density in 1 + 1 space

$$\mathscr{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - V(\phi) \tag{9.129}$$

with

$$V(\phi) = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{1}{4}g(\phi^2 - c^2)^2 - B(\phi - \phi_+). \tag{9.130}$$

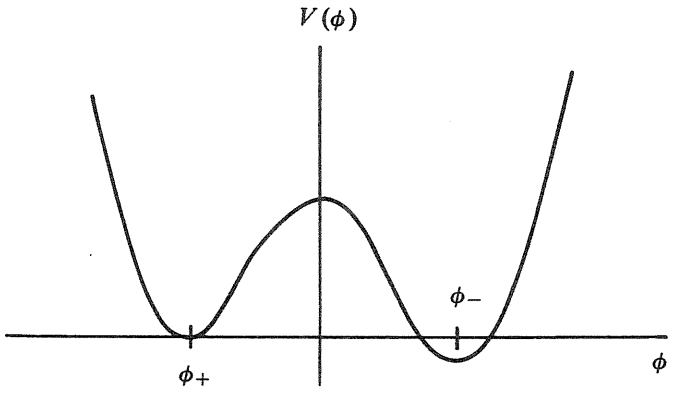


Fig. 9.7 Potential energy density as a function of constant ϕ.

At $B = 0$, we have two degenerate classical vacuums described by $\phi = \pm c$. For a small and positive B, the potential energy density $V(\phi)$ has two minima at

$$\phi_- \approx c, \quad \phi_+ \approx -c \tag{9.131}$$

with the energy density difference

$$\varepsilon \equiv V(\phi_+) - V(\phi_-) \approx 2Bc. \tag{9.132}$$

The classical stable state at $\phi = \phi_+$ is no longer a true ground state, and is referred to as a false vacuum. When a system is cooling down from a high temperature to a low temperature state, it may be trapped, at least locally, in the false vacuum configuration. It is important to know how the false

vacuum state decays into the true vacuum state. This phenomenon is important to some cosmology models.

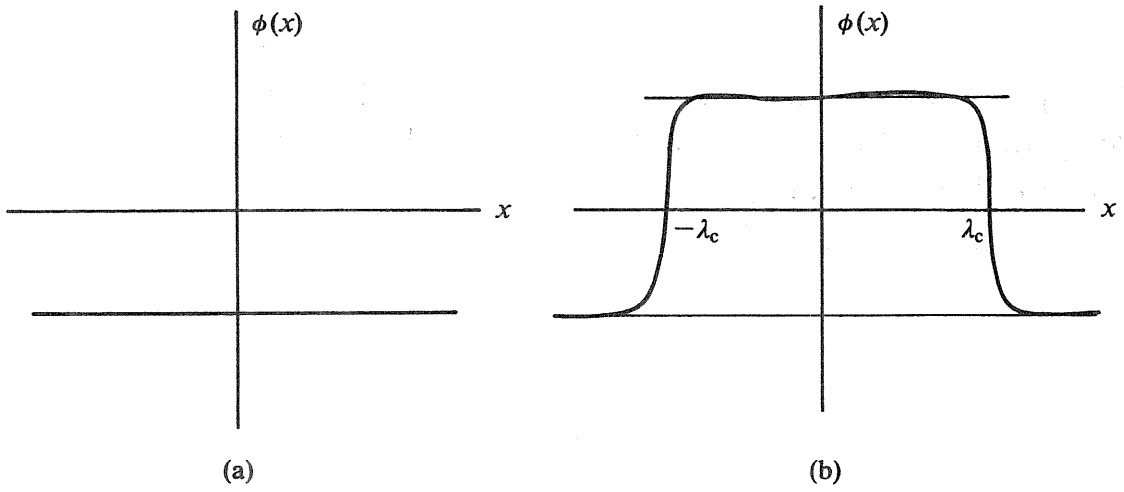


Fig. 9.8 (a) The false vacuum state, and (b) the false vacuum state plus a kink-antikink pair.

To initiate a decay, a false vacuum state only needs to tunnel into a false vacuum plus a kink-antikink pair. If the kink and the antikink are separated farther enough, the energy gained by forming a segment of true vacuum can overcome the energy investment of creating the kink and antikink masses. The kink and the antikink configurations located at $x = \pm\lambda_c$ are approximately

$$\phi_\kappa = c \tanh \frac{\mu}{2}(x + \lambda_c) \tag{9.133a}$$

$$\phi_{\bar{\kappa}} = -c \tanh \frac{\mu}{2}(x - \lambda_c) \tag{9.133b}$$

with

$$\mu = (2gc^2)^{1/2}. \tag{9.134}$$

The kink mass is

$$M = \frac{2\sqrt{2}}{3} g^{1/2} c^3. \tag{9.135}$$

The total energy of the kink-antikink system in Fig. 9.8b is

$$E = 2M - 2\varepsilon\lambda_c, \tag{9.136}$$

which is zero at a critical separation

$$\lambda_c = M/\varepsilon. \tag{9.137}$$

Once produced, the kink and the antikink will soon move away with approximately the speed of light, and leave behind them an increasing segment of true vacuum.

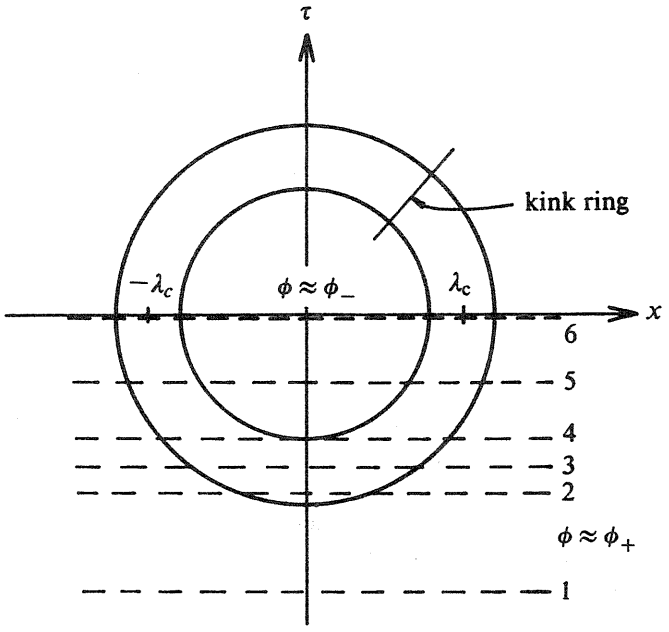


Fig. 9.9 The Euclidean solution associated with the decay of a false-vacuum.

To find the tunnelling configuration, we look at a Euclidean solution to

$$\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial x^2} + g(\phi^2 - c^2)\,\phi - B = 0. \tag{9.138}$$

The solution shown in Fig. 9.9 is a circular symmetric solution in the $x\tau$ plane, with $\phi = \phi_+$ as $r =$ infinity, and $\phi = \phi_-$ at the origin ($r = 0$). The interface of these two regions is a ring of kink configuration. The false vacuum state corresponds to the configuration at $\tau = -\infty$, and the kink-antikink state at $x = \pm\lambda_c$ corresponds to $\tau = 0$ in the $x\tau$ plane. The Euclidean solution is given approximately by

$$\phi(x, \tau) = -c \tanh\frac{\mu}{2}(r - \lambda_c). \tag{9.139}$$

The Euclidean section associated with this solution is

$$\int_{-\infty}^{0} d\tau \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{g}{4}(\phi^2 - c^2)^2 - B(\phi - \phi_+) \right]$$

$$= \tau \lambda_c M - \frac{1}{2}\pi \lambda_c^2 \varepsilon \tag{9.140a}$$

$$= \pi M^2/2\varepsilon \tag{9.140b}$$

where in (9.140a) the first term is associated with the contribution due to the ring, and the second term is associated with the lowering of the energy density in the true vacuum. Thus, the WKB tunnelling amplitude for the decay of a false vacuum state is exp $(-\pi M^2/2\varepsilon)$.

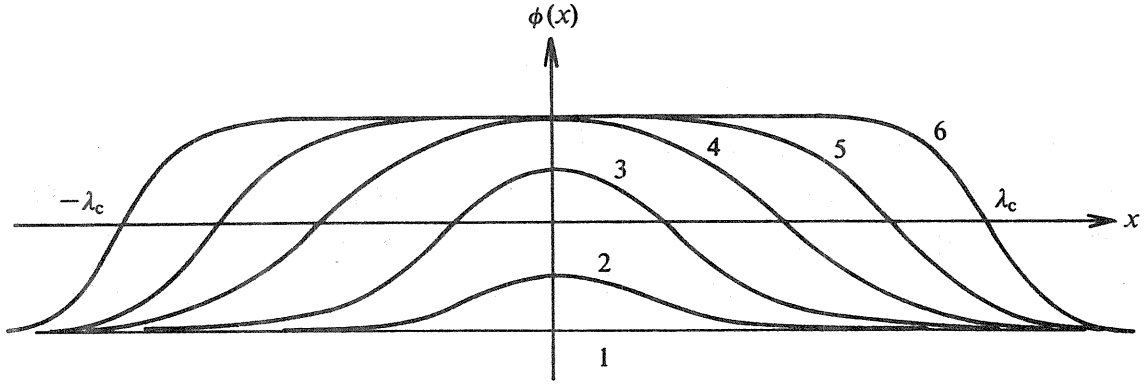


Fig. 9.10 Family of configurations associated with the MPEP. Labels 1–6 describe the cross sections in Fig. 9.9.

It is useful to consider the MPEP associated with this tunnelling process. As we follow the Euclidean solution $\phi(x, \tau)$ for different τ's, we obtain a family of configurations as shown in Fig. 9.10. By fluctuation, a virtual kink-antikink pair is formed (See configuration 1–4 in Fig. 9.10). The pair then moves outward until it becomes a real kink-antikink pair at $\pm\lambda_c$. The virtual kink and antikink at $\pm\lambda(\lambda < \lambda_c)$ have the x-dependence as

$$\phi_K = c \tanh\left(\frac{\mu}{2}(x + \lambda)\frac{\lambda}{\lambda_c}\right) \tag{9.141a}$$

$$\phi_{\bar{K}} = -c \tanh\left(\frac{\mu}{2}(x - \lambda)\frac{\lambda}{\lambda_c}\right). \tag{9.141b}$$

When we treat the kink and the antikink as relativistic particles, we find

that their total energy always obeys

$$E = \frac{2M}{\sqrt{1-\dot{\lambda}^2}} - 2\varepsilon\lambda = 0. \tag{9.142}$$

For $\lambda > \lambda_c$, the kink and antikink are real, and we have

$$\frac{1}{\sqrt{1-\dot{\lambda}^2}} = \frac{\varepsilon\lambda}{M} = \frac{\lambda}{\lambda_c} > 1. \tag{9.143}$$

The λ/λ_c in (9.145) would give the Lorentz contraction factor. For $\lambda < \lambda_c$, the kink and antikink are virtual, and

$$\frac{1}{\sqrt{1-\dot{\lambda}^2}} = \frac{\lambda}{\lambda_c} < 1$$

describes a "Lorentz dilation". The virtual kink and antikink have widths wider than those of real particles. The width varies continuously when the kink (antikink) changes from a virtual particle to a real particle.

We may extend the above method to 3+1 dimensions straightforwardly.

SUBJECT INDEX

action principle, 187,188,191,196,212
adjoint representation, 302
angular-momentum operators, 203,206,211
annihilation operators, 9,31,79,121,122,172,200
anticommutators, 30
antilinear operator, 223
antiparticle, 31
antiunitary operator, 202,223,224
asymptotic freedom, 341
auxiliary conditions, 196
axial
  anomaly, 364
  vector, 26
    field, 219

bare
  charge, 106,286
  mass, 110
Bethe-Salpeter equation, 147
Bhabha amplitude, 88
Bianchi identities, 271
bilinear form, 210,220,223,225,226
Bloch-wave, 357
Bogoliubov transformation, 164
Boltzmann
  distribution, 183
  factor, 167
bose condensation, 153
Bose statistics, 227
boson fermion, 38
BPHZ renormalization, 292
Brownian motion, 181
Bruekner theory, 146

Callan-Symanzik equation, 326,330
canonical
  equal-time quantization relations, 122
  quantization relations, 15,37,188,195,208,224
Casimir operators, 310,339
causality, 229
charge conjugation, 20,26,220
  transformation, 221
charged field, 221
chemical potential, 153
classical
  limit, 232
  system with noise, 172
coherent state, 92
completeness conditions, 13,29
Compton scattering, 95
condensate, 156,158
connected vacuum diagrams, 56
constraint equations, 198
contraction, 45
Coulomb
  Green's function, 261
  gauge, 75,77,80,259
creation operators 9,31,79,121,122,172,200
critical velocity, 167
cross sections, 64
current conservation, 83,272

degrees of divergence, 98
dimension regularization, 304
Dirac
  δ-functions, 5
  equation, 20,193
  field, 208
  gamma matrices, 20
  particle, 20
  spinor, 23
discrete Lorentz transformation, 25
disjoint subgraphs, 292
Dyson's
  expansion, 38
  prescriptions, 286
Dyson-Schwinger equation, 247,249,256,257,267
Dyson-Wick's expansion, 178,198,245

Einstein's statistics, 15
electric and magnetic form factors, 119
electromagnetic fields, 74
electron, 20
  gas model, 130
  propagators, 84
  self-energy diagram, 109,281
electroweak, 321
energy operator, 31
energy-momentum operators, 203,205,206
energy-momentum relation, 16
equal-time
  anticommutation relations, 30
canonical quantization relations, 78
  quantization relations, 3,14,16,187,192,197
  quantization rules, 4,19
evolution operator, 40
excitation spectrum, 144,170
exclusion principle, 33,227
external leg, 52

Faddeev-Povpov ghost, 275
false vacuum, 368
Fermi
  energy, 123
  sea, 123
  statistics, 227
fermion, 33
  contraction, 47
  loop, 55
  propagator, 47
Feynman
  parameter representation, 101,116
  path integral, 187
  rules, 62,82,84,124,125,159,160,177,179,198,240,
    246,256,267,272,275,277
field
  quantizations, 187
  variables, 13
Fokker-Planck equation, 182
forest, 295
fountain effect, 166
free Dirac field, 29

functional Schrödinger equation, 6
Furry's theorem, 98,99,127

Galitsdkii's equation, 150
gauge
  group volume, 259,262,273
  invariance, 100
  transformation, 75,83,259,266
Gaussian
  gauge, 263,276,315
  integral, 233
  noise, 172,173,177
Gell-Mann matrices, 301
generating
  function, 190,194,195,197,277
  functional of Green's functions, 243,256,315
generators, 188,191,203
  of an inhomogeneous Lorentz group, 208
  of Lorentz transformations, 208
ghost fields, 275
Goldstone boson, 321
grand unification theory, 303
Grassmann algebra, 250
Green's functions, 173,174,243,277
ground state, 10,121,235
  energy, 57
group volume factor, 266

Hamiltonian, 4,16
  operator, 79
  path integrals, 238
hard core, 145
harmonic oscillator, 233
Hartree approximation, 146
Hartree-Fock approximation, 126,128,131
Heisenberg
  equations, 3,187
  equations of motion are, 122
  operator, 39
  picture, 3,39
  representation, 6
Higg's mechanism, 299,318
hole, 33,123,139
  lines, 147

homogeneous
  Lorentz group, 21
  Lorentz transformation, 23,203

improper self-energy diagrams, 109
in-fields, 42
in-states, 42
infinitesimal generator of the Lie group, 300
infrared divergences, 112
inhomogeneous
  Lorentz group, 204
  Lorentz transformation, 202
insertions, 286
instanton, 343
integrations over Grassmann algebra, 251
interaction
  operator, 39
  picture, 38,39
intrinsic parity, 219
invariant
  amplitude, 61,62
  coupling constant, 331

Jacobi identity, 272,302

Klein-Gordon equation, 7,16,191,228
Klein-Nishina formula, 96
Kronecker δ's, 5

ladder
  amplitudes, 147
  diagrams, 147
Lagrange function, 14,16,29,76,191,193
Lagrangian, 4
λ-point, 165
Legendre transformations, 190
Lie
  algebra, 302
  group, 300
Lorentz
  gauge, 262,272
Lorentz invariance, 20,202,227
Lorentz transformation, 22,202,210
low energy theorem, 97

macroscopic occupation number, 153
Majorana
  representation, 26,221,224
  spinor, 222
many-particle system, 13
mass counter term, 110
Maxwell
  equations are, 74
  field, 258
minimal coupling, 82
modified Hartree-Fock approximation, 150,151
momentum operator, 16,32,80
  space, 61
  Feynman rules, 59,157
  variable, 29
most probable escape path (MPEP), 350,352
Mott cross section formula in, 94

negative energy
  sea, 32,227
  solutions, 26
noise, 172
non-Abelian
  gauge theories, 268,300,304
  theories, 258
non-overlapping graphs, 292
non-relativistic quantum
  mechanics, 230
  mechanical system, 3
non-renormalizable, 297
normal
  fluid, 156,165
  ordering, 44
ordered, 121
    product, 44
nuclear matter, 145
nucleon
  contraction, 49
  force, 146
  self-energy diagrams, 55
  -antinucleon annihilation process, 50
  -nucleon potential, 145

occupation number, 11,153
one-pion exchange processes, 50
out-states, 42
overlapping
  divergences, 290
  insertions, 287

parity
  operator, 218
  transformations, 219
path integral, 232,258,272
Pauli and Villars, 100
Pauli matrices, 20
Pauli-Villars
  method, 304
  scheme, 101
Pfaffian of, 253
phase-space factor, 66,67
phonon, 167
  propagation function, 80
  propagator, 80,82,84
  wave function renormalization constant, 106
  -electron vertices, 84
physical
  charge, 106,286
  mass, 110
π N scattering, 51
π $\bar{N}$ scattering, 51
pion
  contraction, 49
  self-energy diagram, 54
  -nucleon system, 37
Poisson brackets, 187
polar vector field, 219
position space
  Feynman rules, 53,58
  Feynman amplitude, 59
positive energy solutions, 26
positron, 33
projection operators, 28
propagation function, 200
propagator, 45,69,199
proper self-energy diagram, 109,160
Pseudo scalar, 26

field, 219

quantization rule, 187
quantized vortices, 170
quantum
  chromodynamics, 303
  electrodynamics, 83,265,281
quasiparticles, 143

regulation scheme, 100
renormalization, 281,297,308
  constant,
    Z_1, 285
    Z_3, 284
  group, 322,326
    method, 109
  part, 289,293,310
  point, 324,326
renormalized amplitude, 293
representation
  of homogeneous Lorentz group, 204
  of the Lorentz group, 23
rotation group, 21
roton, 167,170
running coupling constant, 108,332

S-matrix, 42,90,201
scalar field, 16,205,240
Schrödinger
  equation, 3,13
  picture, 3
  representation, 5
Schwinger's
  action principle, 187,188,194
  relation, 215
screened Coulomb potential, 137
self-dual, 344
self-energy
  contribution, 51
  diagrams, 160
semi-classical, 343
skeleton 286
  diagram, 286
Slavnov-Taylor identities, 313,317

space inversion, 25
spin and statistics, 227
spin
  multiplicity factor, 153
  projection operator, 77
standard representation, 20
stress tensor, 205,207,209,211
structure constants, 300
subgraph, 292
super-renormalizable, 297
superficial degrees of divergence, 98,293,298
superfluid, 156,165,168
symmetry factor, 58

T^*-product, 201
TCP
  theorem, 226
  transformation, 226
θ vacuum, 353,357
Thomas-Fermi theory, 137
time
  ordering, 44
  reversal, 223
  transformation, 223
  -ordered product, 41,44,47
trace identities, 71
traces over the γ-matrices, 70

U-operator, 40
ultraviolet cutoff, 325
unitarity relation, 144
unitary
  operator, 202
  transformations, 188,202
unrenormalized
  Green's function, 326
  amplitude, 293

vacuum, 12
  diagram, 51,56
  energy density, 57
  polarization diagram, 99,283,305
  state, 17,31
  to vacuum transition amplitude, 241,254,260,263,266,273

tunnelling, 355
vector field, 196
vertex, 49
correction, 112,284

Ward identity, 106,114,285
Ward-Takahashi identities, 315
wave-function renormalization constant, 112
Weinberg-Salam theory, 321
Wick's
rotation, 103
theorem, 43,47,173,243
winding number, 345,357
WKB method, 347

Yang-Mills field, 268
Yang-Mills theory, 272,300

zero
modes, 357
temperature Bose system, 153
temperature fermi system, 121
-point energy, 11
-temperature bose system, 159